ON PREDICTION THEORY FOR NONSTATIONARY SEQUENCES

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1. Introduction

Let H be a Hilbert space with inner product (,) and let x_n, $n = 0, \pm 1, \cdots$, be a (two-sided) sequence of elements of H. Let $B(m, n) = (x_m, x_n)$, and for each n let $H_n(x)$ be the smallest closed (linear) subspace of H containing all the x_m for $m \leq n$. Let $H_{-\infty}(x)$ be the intersection of all the $H_n(x)$, and let $H_{+\infty}(x)$ be the closure of the union of all $H_n(x)$. We call $\{x_n\}$ a *Hilbert sequence.*

A Hilbert sequence $\{x_n\}$ is called *deterministic* if $H_{-\infty}(x) = H_{+\infty}(x)$ and will be called *linearly free* if $H_{-\infty}(x) = 0$. (Some authors call sequences satisfying the latter condition "completely nondeterministic.") Cramér [1] has shown that for any Hilbert sequence $\{x_n\}$ there exist Hilbert sequences $\{u_n\}$ and $\{v_n\}$ with $x_n = u_n + v_n$, $u_n \in H_n(x)$ and $v_n \in H_n(x)$ for all n, $u_m \perp v_n$ for all m and n, $\{u_n\}$ linearly free, and $\{v_n\}$ deterministic ("Wold decomposition").

There is a well worked out theory for stationary sequences, where $B(m, n)$ depends only on $m - n$. Cramér [1] has proved some results for certain classes of nonstationary sequences. Most of these results involve Fourier series or transforms. In recent years, great progress has been made in Fourier analysis by way of L. Schwartz's theory of distributions. However, it appears that distributions have not yet been used much in prediction theory. One paper by Rozanov [5] extends classical results in prediction theory to stationary "random distributions."

In this paper, we find a necessary and sufficient condition that a Hilbert sequence be deterministic in terms of the Fourier transform of its covariance $B(m, n)$, assuming only the following:

$$\text{(1)} \qquad \text{for some polynomial } P,\ B(n, n) \leq P(n) \qquad \text{for all } n.$$

The Fourier transform of $B(m, n)$ will be a Schwartz distribution on the two-dimensional torus (product of two circles).

For stationary sequences, there is a classical criterion for determinacy. Our criterion for nonstationary Hilbert sequences is less satisfactory, since it involves an existence assertion, but it *is* a criterion, and the partial results of Cramér [1] proved under more restrictive hypotheses on B follow fairly easily from it. We also obtain a characterization of the covariances of linearly free sequences.

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