ON THE DENSITIES OF PROBABILITY MEASURES IN FUNCTIONAL SPACES

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1. Introduction

1.1. Let us assume that there is defined on some probability field $\{\Omega, \mathcal{B}, P\}$ a random process $\xi(t, \omega)$, $t \in E$, where E is some set on the line, and $\omega \in \Omega$. We denote by $\mathbf{F}_E$ the set of all functions, defined on the set E and assuming numerical values. The mapping $\xi(\cdot, \omega)$ carries over the σ-algebra $\mathcal{B}$ on Ω to some σ-algebra $\mathfrak{F}$ of subsets of $\mathbf{F}_E$ and the measure P on $\mathcal{B}$ to a measure μ on $\mathfrak{F}$. The σ-algebra $\mathfrak{F}$ contains at least the sets of the form $\{x(\cdot); x(t_1) < x_1\}$ for $t_1 \in E$ and x_1 real (because $\{\omega; \xi(t_1, \omega) < x_1\} \in \mathcal{B}$) and, consequently, contains all cylinder subsets of the space $\mathbf{F}_E$. If we denote by $\mathfrak{F}_0$ the smallest σ-algebra of subsets of $\mathbf{F}_E$ containing all cylinder subsets of $\mathbf{F}_E$, then $\mathfrak{F}_0 \subset \mathfrak{F}$. As a rule the measure μ on $\mathfrak{F}$ is completely determined by its values on $\mathfrak{F}_0$ ($(\mu, \mathfrak{F})$ is the completion of $(\mu, \mathfrak{F}_0)$). Therefore, it suffices to consider the measure μ on the σ-algebra $\mathfrak{F}_0$, which depends only on the set E and not on the specific form of the process. We shall call the measure μ on $\mathfrak{F}_0$ the measure corresponding to the process $\xi(t, \omega)$. In many problems one can identify the process and the measure, because from the measure μ one can define the probability space $\{\mathbf{F}_E, \mathfrak{F}, \mu\}$, on which the natural mapping $\xi(t, x(\cdot)) = x(t)$ defines a random process to which corresponds the measure μ.

If two probability measures μ_1 and μ_2 are defined on the σ-algebra $\mathfrak{F}_0$, then, as is well-known, μ_2 is said to be *absolutely continuous* with respect to μ_1, if $\mu_2(A) = 0$ for all $A \in \mathfrak{F}_0$ for which $\mu_1(A) = 0$. The absolute continuity of μ_2 with respect to μ_1 is a necessary and sufficient condition for the existence of an $\mathfrak{F}_0$-measurable function $\rho(x)$ such that

$$\mu_2(A) = \int_A \rho(x)\mu_1(dx) \tag{1.1}$$

for all $A \in \mathfrak{F}_0$. This function $\rho(x)$ is called the *density* or the *derivative* of the measure μ_2 with respect to μ_1 and is denoted by $(d\mu_2/d\mu_1)(x)$. If, for some A, $\mu_1(A) = 1$, $\mu_2(A) = 0$, then μ_1 and μ_2 are *mutually singular*.

1.2. In recent times a substantial part of the work in the theory of random processes has been devoted to the solution of the question of the absolute continuity (or the singularity) of measures corresponding to random processes. One can indicate various directions, frequently having important practical interest, in which results on the absolute continuity (or singularity) and density