ON FIXED POINTS OF SEMIGROUPS OF ENDOMORPHISMS OF LINEAR SPACES

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1. Introduction

There are several theorems guaranteeing the existence of fixed points of different classes of transformations in various types of spaces. In the case of locally convex linear spaces and families of linear mappings, two of them, due to Markov-Kakutani and Kakutani himself (quoted as theorems 1 and 2 in this paper), are especially prominent.

The main purpose of this paper is to show a generalization (theorem 3) of the second theorem. However, our method of proof is quite different from that of Kakutani. It appears very strange to the author that this theorem which has a "deterministic" content has a "probabilistic" proof, which is based on a kind of Monte Carlo method. Let us mention that this paper is related to an earlier work of the author [6] on random ergodic theorems and their applications. In particular, theorem 5 from [6] was an intermediate form between the theorem of Kakutani and theorem 3 from this paper. Paper [6] is an announcement (some proofs are only sketched there), and the complete proof was never published. Later the author has observed that the most interesting part of [6], namely theorem 5, can be generalized using a more direct method independent of ergodic theory, and this is presented here.

Theorem 3 from this paper gives, of course, the consequences which the earlier theorem 5 from [6] had. For the sake of completeness we reproduce here, as corollaries from theorem 3, the existence of invariant mean for weakly almost periodic functions (which was conjectured, for example, by K. Deleeuw and I. Glicksberg ([1], pp. 88-89)). For other corollaries, see [6].

2. General assumptions and notation

Let G be a semigroup of endormorphisms of a locally convex linear topological space X (that is, every element $T \in G$ is a linear continuous operator from X into X and superposition is the semigroup operation). Let Q be a convex and closed subset of X which is G-*invariant* (namely, $T(Q) \subseteq Q$ for every $T \in G$).