SOME CONTRIBUTIONS TO THE THEORY OF ORDER STATISTICS

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1. Introduction and summary

This paper arose from the problem of proving the asymptotic normality of linear combinations of order statistics which was first posed by Jung [9]. In the course of this investigation, several facts of general interest in the study of moments of order statistics, which either had not been stated or had not been proved in their most satisfactory form, were established. These are collected in theorems 2.1 and 2.2 of section 2. Briefly we show in theorem 2.1 that any two order statistics are positively correlated, and in theorem 2.2 we give necessary and sufficient conditions for the existence of moments of quantiles and the convergence of the suitably normalized moments to those of the appropriate normal distribution.

Section 3 contains an "invariance principle" for order statistics more elementary than the one given by Hájek [7] but requiring fewer regularity conditions and adequate for our purposes in section 4. In an as yet unpublished paper, J. L. Hodges and the author give another application of this principle in deriving the asymptotic distribution of an estimate of location in the one sample problem.

Section 4 contains the principal results of the paper. We consider linear combinations of order statistics which do not involve the extreme statistics to a more significant extent than the sample mean does. For this class of statistics we establish asymptotic normality and convergence of normalized moments to those of the appropriate Gaussian distribution.

2. Some properties of moments of order statistics

Let $X_1, \cdots, X_n$ be a sample from a population with distribution F and density f which is continuous and strictly positive on $\{x|0 < F(x) < 1\}$. Then $F^{-1}(t)$ is well-defined and continuous for $0 < t < 1$, and for those values of t we may define $\psi(t) = f[F^{-1}(t)]$. We denote by $Z_{1,n} < \cdots < Z_{n,n}$ the order statistics of the sample.

The following two theorems will be proved in this section.

THEOREM 2.1. *Suppose that* $E(Z_{i,n}^2) + E(Z_{k,n}^2) < \infty$. *Then,* $\operatorname{cov}(Z_{i,n}, Z_{j,n}) \geq 0$.

THEOREM 2.2. *Suppose that* $\lim_{x\to\infty} |x|^{\epsilon}[1 - F(x) + F(-x)] = 0$ *for some* $\epsilon > 0$. *Then,*

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