EXISTENCE AND PROPERTIES OF CERTAIN OPTIMAL STOPPING RULES

ARYEH DVORETZKY

THE HEBREW UNIVERSITY OF JERUSALEM

and

COLUMBIA UNIVERSITY

1. Introduction

The main purpose of this note is to prove the existence of optimal stopping rules for certain problems involving sums of independent, identically distributed random variables. A special case was treated by Y. S. Chow and H. E. Robbins [2]. Their problem is very easily stated: let s_n be the excess of the number of heads over the number of tails in the first n tosses of an infinite sequence of independent tosses of a fair coin. Does there exist a stopping variable τ for which the expected average gain is maximal? In other words, does there exist a τ for which the expectation of s_τ/τ is at least as great as the expectation of s_t/t for any other stopping variable t? It turns out that this simple problem is not reducible to any of the available standard results on the existence of optimal stopping rules. Chow and Robbins do prove the existence of an optimal τ by an ingenious method which is, at least in part, suited only to the special case which they consider. Here, following in part the method of [2] and substituting general considerations for the specific ones used there, we establish the existence of an optimal stopping variable, maximizing the expected average gain under the sole assumption that the random variables involved have finite variance.

In section 2 we prove the above result. Our method also yields interesting information on the structure of the optimal τ which we present in section 3. For the sake of clarity, we confined the main exposition to the problem of maximizing the expected average gain; however, the methods developed here can deal with more general situations, and one generalization is presented in section 4. The last section contains various remarks.

Throughout we denote by $(\Omega, \mathfrak{B}, P)$ the underlying probability space. Also, E denotes expectation, and we write $\{\cdots\}$ to denote $\{\omega: \cdots\}$, the set of ω having the indicated properties. All random variables are, of course, defined only almost surely but, in the interest of brevity, this qualification is usually omitted.

Throughout the paper, $x_1, x_2, \cdots, x_n, \cdots$, is a sequence of *independent, iden-*