MOMENTS OF CHI AND POWER OF t

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1. Introduction and summary

This paper is concerned with two related computational problems: the precise calculation of central moments of the chi random variable of ν degrees of freedom, and the use of these moments in computing the power curve of the t-test. Whereas the methods are standard and available in various textbooks, the results have at several points been pushed farther than we have seen them elsewhere. We try to provide the formulas and coefficient tables that would be needed by the computer, but make no attempt to review the extensive literature on chi moments and t power.

Table A gives the coefficients required for obtaining the first twelve moments in terms of the expectation ϵ_ν. In section 3 the general term of an asymptotic series for log ϵ_ν is derived, which provides in table C the early coefficients of the series for ϵ_ν itself. Section 5 presents a formula for the first three terms in the series for moments of arbitrary order, supplemented in table E by additional terms for the first twelve moments. With these coefficients it is relatively easy to obtain precise values of the low moments for large ν.

Section 6 presents a series for the power of the t test in terms of chi moments. In favorable cases this method permits the precise computation of an entire power curve. It also leads to a relatively simple normal approximation for t power, accurate when ν is not too small and the significance level is moderate, and suggests an effective method of interpolation in the noncentral t tables.

2. The moments of chi in terms of its expectation

Let χ denote the chi random variable with ν degrees of freedom, and consider its standardized form $S = \chi/\sqrt{\nu}$. It is well known that

$$ES^p = \Gamma\left(\frac{\nu + p}{2}\right) \Big/ \left(\frac{\nu}{2}\right)^{p/2} \Gamma\left(\frac{\nu}{2}\right), \qquad p = 1, 2, \cdots, \tag{2.1}$$

and that both the original and central moments of S can be expressed in terms of its expectation,

$$ES = \epsilon_\nu = \Gamma\left(\frac{\nu + 1}{2}\right) \Big/ \sqrt{\frac{\nu}{2}}\, \Gamma\left(\frac{\nu}{2}\right). \tag{2.2}$$

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