AN APPLICATION OF ERGODIC THEOREMS IN THE THEORY OF QUEUES

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1. Introduction

We establish some notation: $\{Z_k, k \geqq 0\}$ is a homogeneous Markov chain taking its values in a locally compact Hausdorff space $\mathfrak{X}$; we denote by Σ the σ-algebra of subsets of $\mathfrak{X}$ generated by the open sets; $ca(\Sigma)$ is the Banach space of totally finite regular measures on Σ; and $C_0(\mathfrak{X})$ is the Banach space of real-valued, bounded, continuous functions on $\mathfrak{X}$ which vanish at infinity. If $\Phi_k \in ca(\Sigma)$ is the probability measure of Z_k then there exists a bounded linear operator T on $ca(\Sigma)$ into itself such that $\Phi_{k+1} = T\Phi_k$. If this operator can be represented by a real-valued function P on the product space $\mathfrak{X} \times \Sigma$ with the properties

(a) $0 \leqq P(x, F) \leqq P(x, \mathfrak{X}) = 1$ for all $x \in \mathfrak{X}$, $F \in \Sigma$;
(b) for each $x \in \mathfrak{X}$, $P(x, \cdot) \in ca(\Sigma)$;
(c) for each $F \in \Sigma$, $P(\cdot, F)$ is Σ-measurable;

then the mapping of $ca(\Sigma)$ into itself takes the form

$$\Phi_{k+1}(F) = \int_{\mathfrak{X}} \Phi_k(dx)P(x, F) \tag{1}$$

for each $F \in \Sigma$. We define inductively a sequence of real-valued functions $P_r(\cdot, \cdot)$ on $\mathfrak{X} \times \Sigma$ by the relations

$$\begin{aligned} P_{r+1}(x, F) &= \int_{\mathfrak{X}} P_r(x, dy)\, P_1(y, F) \\ P_1(x, F) &\equiv P(x, F). \end{aligned} \tag{2}$$

We may identify the conditional probability $P\{Z_{k+r} \in F | Z_k = x\}$ with the function $P_r(x, F)$, so that the rth iterate of the operator T may be written

$$(T^r\Phi)(F) = \int_{\mathfrak{X}} \Phi(dx)\, P_r(x, F). \tag{3}$$

A principal problem of ergodic theory has been to determine conditions under which the sequence of operators $n^{-1} \sum_{r=0}^{n-1} T^r$ converges in some sense. The