RECURRENT RANDOM WALK AND LOGARITHMIC POTENTIAL

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1. Introduction

This is an attempt to show that a potential theory is associated with certain recurrent Markov processes in a natural way. For transient Markov processes this fact has been studied intensely. Thus Hunt [9] bases a general potential theory on transient continuous parameter processes, Doob [7] and Hunt [10] use the theory to construct boundaries for discrete parameter processes, Itô and McKean [11] solve the problem of characterizing the recurrent sets for simple random walk in three and higher dimension within the framework of the associated potential theory.

The class of recurrent Markov processes considered here are defined as follows. The state space will be E, the set of all ordered pairs (lattice points) $x = (m, n)$ where m and n are integers. X is a random variable with values in E. Its characteristic function $\phi(\theta)$ is

$$\phi(\theta) = E[e^{i\theta\cdot X}] = \sum_{x \in E} e^{i\theta\cdot x} P\{X = x\}, \tag{1.1}$$

where $\theta = (\theta_1, \theta_2)$ and $\theta\cdot x = \theta_1 m + \theta_2 n$ if $x = (m, n)$. Here $X_1, X_2, \cdots$ is an infinite sequence of independent random variables with the same distribution as X. Each characteristic function $\phi(\theta)$ defines a Markov process

$$S_n = S_0 + X_1 + \cdots + X_n, \qquad n \geqq 1, \tag{1.2}$$

where the starting point S_0 is an arbitrary point in E.

For simplicity we assume throughout that $\phi(\theta) = \phi(-\theta)$, or equivalently that $P\{X = x\} = P\{X = -x\}$. Two further assumptions are essential to the theory. Let S denote the square $|\theta_1| \leqq \pi$, $|\theta_2| \leqq \pi$ in the θ-plane and let $\int [\ ]\, d\theta$ denote integration over S where $d\theta$ is two-dimensional Lebesgue measure. For $\theta \in S$, it is assumed that $\phi(\theta)$ satisfies

$$\phi(\theta) = 1 \Rightarrow \theta = 0 \tag{1.3}$$

$$\int \frac{d\theta}{1 - \phi(\theta)} = \infty. \tag{1.4}$$

A number theoretical argument may be used to show that (1.3) is equivalent to the condition that every x in E is a possible value of the process S_n, that is,