ON SOME GENERAL RENEWAL THEOREMS FOR NONIDENTICALLY DISTRIBUTED VARIABLES

WALTER L. SMITH
UNIVERSITY OF NORTH CAROLINA

1. Introduction

As a convenience, let us agree to call an infinite sequence $X_n \equiv X_1, X_2, X_3, \cdots,$ of independent random variables, a *renewal sequence*, and when all the random variables are identically distributed let us call $\{X_n\}$ a *renewal process*. If all the random variables are nonnegative let us say $\{X_n\}$ is a *positive* renewal sequence (process).

The renewal sequence (process) will be called *periodic* if there is a real $\varpi > 0$ such that, with probability one, every random variable in the renewal sequence (process) is a multiple of ϖ. If the renewal sequence (process) is not periodic we shall call it *continuous*.

We shall write $S_n = X_1 + X_2 + \cdots = X_n$ with $n = 1, 2, 3, \cdots,$ for the partial sums of the renewal sequence, $F_n(x) = P\{S_n \leqq x\}$ for the distribution function of S_n, and $U(x) = P\{0 \leqq x\}$ for the so-called Heaviside unit function. We then define the random variable $N(x)$ as the number of partial sums S_n which satisfy the inequality $S_n \leqq x$,

$$N(x) = \sum_{j=1}^{\infty} U(x - S_j). \tag{1.1}$$

Thus, if $H(x) = E\{N(x)\}$, it follows from (1.1) that

$$H(x) = \sum_{j=1}^{\infty} F_j(x). \tag{1.2}$$

The function $H(x)$ is called the *renewal function* and is of prime interest in renewal theory; we refer to Smith [11] for an extensive account of it. A knowledge of its asymptotic behavior has proved very useful in establishing a variety of results about stochastic processes. However, with very few exceptions (one being the paper by Cox and Smith [5]) almost all the work done so far in this subject has referred to renewal processes (possibly with the trivial modification of allowing X_1 a different distribution from all the other X_n for $n > 1$) rather than the more general renewal sequences.

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