SOME THEOREMS ON CHARACTERISTIC FUNCTIONS OF PROBABILITY DISTRIBUTIONS

E. J. G. PITMAN
UNIVERSITY OF TASMANIA

1. Introduction

Let X be a real valued random variable with probability measure P and distribution function F. It will be convenient to take F as the *intermediate* distribution function defined by

$$F(x) = \frac{1}{2}[P\{X < x\} + P\{X \leqq x\}]. \tag{1.1}$$

In mathematical analysis it is a little more convenient to use this function rather than

$$F_1(x) = P\{X < x\} \quad \text{or} \quad F_2(x) = P\{X \leqq x), \tag{1.2}$$

which arise more naturally in probability theory. With this definition, if the distribution function of X is $F(x)$, then the distribution function of $-X$ is $1 - F(-x)$. The distribution of X is symmetrical about 0 if $F(x) = 1 - F(-x)$. For F_1 and F_2 the corresponding relations are more complicated at points of discontinuity.

The characteristic function of X, or of F, is

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x), \tag{1.3}$$

defined and uniformly continuous for all real t. The function ϕ is uniquely determined by F. Conversely, F is uniquely determined by ϕ. Every property of F must be implicit in ϕ and vice versa. It is often an interesting but difficult problem to determine what property of one function corresponds to a specified property of its transform.

We know that in a general way the behavior of $F(x)$ for large x is related to the behavior of $\phi(t)$ in the neighborhood of $t = 0$. The main object of this paper is to make some precise and rather simple statements about this relation. We are interested in the behavior of $\phi(t)$ in the neighborhood of $t = 0$ because upon this depend all limit theorems on sums of random variables. For example, suppose that $X_1, X_2, \cdots$ is a sequence of independent, identically distributed random variables with distribution function $F(x)$ and characteristic function