AN EXTENSION OF THE LEBESGUE MEASURE OF LINEAR SETS

PAUL LÉVY
ÉCOLE POLYTECHNIQUE

1. Introduction

The present paper develops the following idea of which I told Lebesgue almost forty years ago. It is possible to define on the circle of length unity a set Γ such that the circle is the union of a countable infinity of sets Γ_k which are disjoint and superposable on Γ by rotation. If K_h is the union of those Γ_k for which $k \equiv h \bmod p$, where $p > 1$ is an integer, the circle is also the union of p disjoint sets K_h which are superposable by rotation. For an extension of the Lebesgue measure invariant under rotation the Γ_k cannot be measurable. However, the K_h can be measurable, their common measure being $1/p$.

After a brief preliminary section (section 2), these results will be established in section 3.

Lebesgue, who was not interested in arguments depending on Zermelo's axiom of choice, led me away from a further study of these sets. Besides, I was soon to learn that the sets Γ had already been defined by Vitali. I do not know whether the sets K had also attracted his attention.

The reason for my present return to these problems is the remark that my old results can be substantially improved by using G. Hamel's theorem according to which the real numbers possess a basis $\{\omega_\nu\}$. Every real number x is then uniquely defined by a noncountable family of *rational components* a_ν. Restricting one of the a_ν to be an element of the interval $[0, 1)$, one obtains a set of the Vitali type. From this it is possible to construct sets K_ν^h which are easily made measurable by a suitable extension of the Lebesgue measure. The application of the same procedure to any finite set of components a_ν yields a new extension.

An extension of the Lebesgue measure which reaches considerably further can be obtained through the use of two Hamel bases $\{\omega_\nu\}$ and $\{\omega'_\nu\}$. Consider then the numbers x and $y = f(x)$ which correspond for these two bases to the same rational components a_ν. If $f(1) = 1$, as will be assumed here, the fractional part $g(x)$ of $f(x)$, defined for $x \in [0, 1)$, is a one-to-one function of x. Suppose that, in the square $[0, 1) \times [0, 1)$, the complement of the graph $\mathcal{G}$ of the function g has interior measure zero. If E is a measurable subset of the square having Lebesgue measure $m(E)$ and if $\mathcal{E}$ denotes the set of values of x for which $[x, g(x)] \in E$, then $\mu^*(\mathcal{E}) = m(E)$ defines an extension of the Lebesgue measure.