OCCUPATION TIME LAWS FOR BIRTH AND DEATH PROCESSES

SAMUEL KARLIN AND JAMES McGREGOR
STANFORD UNIVERSITY

1. Introduction

Let $X(t)$ with $t \geqq 0$ be a (Borel) measurable stationary Markov process whose state space is a metric space $\mathcal{E}$ and whose transition probability function is

$$P(t; x, E) = P\{X(t+s) \in E | X(s) = x\}. \tag{1}$$

Darling and Kac [3] studied the limiting distribution of the random variables

$$Z(t) = \int_0^t V[X(\tau)]\, d\tau \tag{2}$$

as $t \to \infty$, where V is a nonnegative measurable function. If V is the characteristic function of a set E then $Z(t)$ is the occupation time of E.

Darling and Kac assume that

$$\int_0^\infty e^{-st} P(t; x, E)\, dt = \pi(E) h(s) + h_1(s; x, E), \tag{3}$$

where $h(s) \to \infty$ as $s \to 0+$ and

$$\lim_{s \to 0+} \frac{1}{h(s)} \int h_1(s; x, dy) V(y) = 0, \tag{4}$$

the convergence in (4) being uniform on the set $\{x; V(x) > 0\}$. They then show that the kth moment $\mu_k(t)$ of $Z(t)$ satisfies

$$s \int_0^\infty e^{-st} \mu_k(t)\, dt \sim k! [\pi(E) h(s)]^k, \qquad s \to 0+. \tag{5}$$

Under the additional hypothesis

$$h(s) = s^{-\alpha} L\left(\frac{1}{s}\right), \tag{6}$$

where $0 \leqq \alpha < 1$ and $L(1/s)$ is slowly varying (see below) as $s \to 0+$, they deduce from Karamata's Tauberian theorem that

$$\lim_{t \to \infty} P\left\{\frac{Z(t)}{\pi(E) h(1/t)} \leqq u\right\} = G_\alpha(u), \tag{7}$$

where G_α denotes the Mittag-Leffler distribution,

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