SPECTRAL ANALYSIS OF STATIONARY GAUSSIAN PROCESSES

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1. Gauss functions

Let $(\Omega, \mathcal{E}, P)$ be a probability space, that is, $\Omega = \{\omega\}$ is a set of elements ω, and $\mathcal{E} = \{E\}$ is a sigma field of subsets E of Ω, and $P(E)$ is a countably additive measure defined on $\mathcal{E}$ with $P(\Omega) = 1$. We denote by $L^2(\Omega)$ the real L^2-space over $(\Omega, \mathcal{E}, P)$, that is, the real linear space of all real-valued $\mathcal{E}$-measurable functions $f(\omega)$ defined on Ω such that

$$(1) \qquad \|f\|^2 = \int_\Omega |f(\omega)|^2 P(d\omega) < \infty.$$

Two functions from $L^2(\Omega)$ which coincide almost everywhere on Ω are identified in $L^2(\Omega)$. For any two functions $f(\omega)$ and $g(\omega)$ from $L^2(\Omega)$, their inner product (f, g) is defined by

$$(2) \qquad (f, g) = \int_\Omega f(\omega) g(\omega) P(d\omega).$$

A function $x(\omega)$ from $L^2(\Omega)$ is called a *Gauss function* if either (i) $x(\omega) = 0$ almost everywhere on Ω, or (ii) there exists a positive number $\sigma > 0$ such that

$$(3) \qquad P\{\omega | \alpha < x(\omega) < \beta\} = \frac{1}{\sqrt{2\pi\sigma}} \int_\alpha^\beta \exp\left(-\frac{u^2}{2\sigma}\right) du$$

for any real numbers α and β with $\alpha < \beta$. In the second case (ii), the function $x(\omega)$ is said to have a *Gaussian distribution* with mean 0 and variance $\sigma > 0$.

2. Gauss systems

Let $\mathcal{G} = \{x_1(\omega), \cdots, x_n(\omega)\}$ be a finite set of functions from $L^2(\Omega)$. Then $\mathcal{G}$ is called a *Gauss system* if the linear combination $\sum_{k=1}^n c_k x_k(\omega)$ is a Gauss function for any real numbers $c_1, \cdots, c_n$. Further $\mathcal{G}$ is said to have an *n-dimensional Gaussian distribution* with mean 0 if there exists a real positive definite matrix $A = (a_{k,l} | k, l = 1, \cdots, n)$ such that

$$(4) \qquad P\{\omega | \alpha_k < x_k(\omega) < \beta_k, k = 1, \cdots, n\}$$
$$= \left(\frac{\det A}{(2\pi)^n}\right)^{1/2} \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_n}^{\beta_n} \exp\left[-\frac{1}{2}(Au, u)\right] du,$$