WIENER INTEGRAL AND FEYNMAN INTEGRAL

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1. Introduction

Consider, for example, a classical mechanical system with Lagrangian

$$L(x, \dot{x}) = \frac{\dot{x}^2}{2} - U(x). \tag{1.1}$$

The wave function of the quantum mechanical system corresponding to this classical one changes with time t according to the Schrödinger equation

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = \frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial x^2} - U\psi, \qquad \psi(0+, x) = \varphi(x). \tag{1.2}$$

Feynman [3] expressed this wave function $\psi(t, x)$ in the following integral form, which we shall here call the *Feynman integral*

$$\psi(t, x) = \frac{1}{N}\int_{\Gamma_x} \exp\left\{\frac{i}{\hbar}\int_0^t \left[\frac{\dot{x}_\tau^2}{2} - U(x_\tau)\right] d\tau\right\} \varphi(x_t) \prod_\tau dx_\tau, \tag{1.3}$$

where Γ_x is the space of paths $X = (x_\tau, 0 < \tau \leqq t)$ with $x_0 = x$, $\prod_\tau dx_\tau$ is a uniform measure on $R^{(0,t]}$, and N is a normalization factor. It should be noted that the integral $\int_0^t [\dot{x}_\tau^2/2 - U(x_\tau)]\, d\tau$ is the classical action integral along the path X. (This idea goes back to Dirac [1].) It is easy to see that (1.3) solves (1.2) unless we require mathematical rigor. It is our purpose to define the generalized measure $\prod_\tau dx_\tau/N$, that is, the integral $\int_{\Gamma_x} F(X)\prod_\tau dx_\tau/N$, rigorously and to prove that (1.3) solves (1.2) in case $U(x) \equiv 0$ (case of no force) or $U(x) \equiv x$ (case of constant force). See theorem 5.2 and theorem 5.3 below. We hope this fact will be proved for a general $U(x)$ with some appropriate regularity conditions.

Our definition is also applicable to the *Wiener integral*; namely, using it, we shall prove that the solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - Uu, \qquad u(0+, x) = f(x), \tag{1.4}$$

is given by

$$u(t, x) = \frac{1}{N}\int_{\Gamma_x} \exp\left\{-\int_0^t \left[\frac{\dot{x}^2}{2} + U(x_\tau)\right] d\tau\right\} f(x_t) \prod_\tau dx_\tau \tag{1.5}$$