ON SEQUENCES OF SUMS OF INDEPENDENT RANDOM VECTORS

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1. Introduction and summary

This paper is concerned with certain properties of the sequence $S_1, S_2, \cdots$ of the sums $S_n = X_1 + \cdots + X_n$ of independent, identically distributed, k-dimensional random vectors $X_1, X_2, \cdots$, where $k \geqq 1$. Attention is restricted to vectors X_n with integer-valued components. Let $A_1, A_2, \cdots$ be a sequence of k-dimensional measurable sets and let N denote the least n for which $S_n \in A_n$. The values $S_0 = 0, S_1, S_2, \cdots$ may be thought of as the successive positions of a moving particle which starts at the origin. The particle is absorbed when it enters set A_n at time n, and N is the time at which absorption occurs. Let M denote the number of times the particle is at the origin prior to absorption (the number of integers n, where $0 \leqq n < N$, for which $S_n = 0$). For the special case $P\{X_n = -1\} = P\{X_n = 1\} = 1/2$ it is found that

$$E(M) = E(|S_N|) \tag{1.1}$$

whenever $E(N) < \infty$. Thus the expected number of times the particle is at the origin prior to absorption equals its expected distance from the origin at the moment of absorption, for any time-dependent absorption boundary such that the expected time of absorption is finite. Some restriction like $E(N) < \infty$ is essential. Indeed, if N is the least $n \geqq 1$ such that $S_n = 0$, equation (1.1) would imply $1 = 0$. In this case $E(N) = \infty$.

The primary concern of this paper is to show that a result analogous to equation (1.1) is true for one-dimensional random variables under rather general conditions, and to obtain a similar result in two dimensions. The proof of equation (1.1) and its generalizations is based on an extension by Blackwell and Girshick [1] of an equation of Wald, the following special case of which is used (see theorem 2.1). If X_n is k-dimensional with $E(|X_n|) < \infty$, where, for $a = (a_1, \cdots, a_k)$, $|a| = (a_1^2 + \cdots + a_k^2)^{1/2}$, and $E(N) < \infty$, then

$$E(M) = E[g(S_N)], \tag{1.2}$$

where $g(s)$ is a solution of the equation

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