STOCHASTIC GROUPS AND RELATED STRUCTURES

ULF GRENANDER
UNIVERSITY OF STOCKHOLM

1. Introduction

The last few years have witnessed an increasing interest in the probability theory of general algebraic and topological structures, especially for topological groups and linear vector spaces. This paper attempts to survey this new field and to present some of the main results together, so as to obtain as complete an exposition as possible of the present state of development. As will be painfully obvious in the following pages no unified theory exists yet, and we can answer only partially the problems that arise. Still, when the existing results are viewed together, it is hoped that the overall picture will be suggestive.

To avoid excessive length and the obscuring of general ideas by details, no proofs are given. To compensate for this much attention is given to the description of the analytical tools that are suitable for proving the sort of results that are described in the following.

The generality of the subject might give the impression that these are abstract and rather vague problems. Actually the situation is just the opposite: this is a piece of very concrete mathematics and many of the problems can be phrased in simple and direct form, although they may be far from simple to solve. The fundamental character of the questions makes this extension of classical probability theory a fascinating study for the probabilist and analyst in search of nontrivial generalizations of classical probability theory.

2. Semigroups

Consider a Hausdorff space S with a binary operation xy on $x, y \in s$, such that S is a topological semigroup. From the subsets of S we form the σ-algebra $\mathcal{B}(S)$ of Borel sets and the set $\mathcal{P}(S)$ of regular probability measure defined on $\mathcal{B}(S)$.

For two measures $P_1, P_2 \in \mathcal{P}(S)$ we define their convolution

$$P_1 * P_2(E) = \int_{xy \in E} dP_1 \times dP_2\,(x, y), \tag{2.1}$$

where $E \in \mathcal{B}(S)$ and the product measure $P_1 \times P_2$ on $S \times S$ is introduced in