NONINCREASE EVERYWHERE OF THE BROWNIAN MOTION PROCESS

A. DVORETZKY
COLUMBIA UNIVERSITY
P. ERDÖS
HEBREW UNIVERSITY OF JERUSALEM
and
S. KAKUTANI
YALE UNIVERSITY

1. Introduction

The (linear, separable) Brownian motion process has been studied more than any other stochastic process. It has many applications and, at least since Bachelier, probabilists have been attracted by its delicate and curious properties. It furnished, in the hands of N. Wiener, the first instance of a satisfactorily defined nondiscrete stochastic process with continuous time parameter, and it is this work on Brownian motion (also known as Wiener space) that suggested the, now universally adopted, method of A. N. Kolmogorov for defining stochastic processes. Most advanced books on probability devote some space to this process but the more delicate results are beyond their scope. A notable exception is P. Lévy [2] which contains a very profound study of the process. However, though the proof of our principal result could be expedited by appealing to some advanced work on Brownian motion we preferred a presentation using only the simpler and better known properties of the process.

The Brownian motion process can be described as a probability space whose elements are all continuous functions defined on the whole real line and vanishing at the origin. The principal aim of this paper is to prove the, to us rather unexpected, result that *the probability of the set of functions which increase at least at one point is zero.* [A function is said to increase at a point if its values slightly to the right (left) of this point are not smaller (larger) than its value at the point.]

A formal statement of this result will be given in the next section and its significance will be discussed in the following one. Section 4 will give an interesting, though wrong and leading to a wrong result, heuristic argument. The

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