NOTES ON MARTINGALE THEORY

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1. Introduction

Although several writers, for example Bernstein, Lévy, and Ville, had used what would now be identified as martingale concepts, the first systematic studies appeared in [2] and [3]. Since then, martingale theory has been applied extensively, but little progress has been made in the theory itself. The purpose of the present paper is to point out how much spade work remains to be done in the theory, by deriving new theorems without the use of deep technical apparatus.

Throughout this paper, the more appropriate nomenclature "submartingale," "supermartingale" is used, rather than the "semimartingale," "lower semimartingale" found in [3]. The unifying thread in the following work will be the fact that certain simple operations on submartingales transform them into submartingales. This leads to a new submartingale convergence theorem, to a sharpening of the upcrossing inequality, and thereby into an examination of apparently hitherto unnoticed interrelations between martingale and potential theory.

2. A new submartingale convergence theorem

The theorems about derivatives of set functions on nets led Chow [1] to study submartingales relative to atomic fields, and he deduced a new submartingale convergence theorem. Theorem 2.1 generalizes Chow's result, and shows how it can be made to depend on the fact that certain transformations take submartingales into submartingales.

We first prove a lemma. The point of this lemma is that, although the sample sequences of a submartingale increase on the average, they may increase more than is necessary to preserve the submartingale property. It may therefore be possible to cut down the random variables of a submartingale the first time a given barrier is passed, and thereby to obtain a submartingale which is bounded from above. The condition (2.1) of the lemma is unusual in that, instead of restricting the excess of the sample sequence the first time the barrier is crossed, as is customary, the probability of crossing is supposed not too small.

LEMMA 2.1. *Let* $\{x_n, \mathfrak{F}_n, n \geqq 1\}$ *be a submartingale, and let* δ, a, b *be specified constants, with* $a \leqq b$, $\delta > 0$. *Suppose that* $\Lambda_0 = \emptyset$, *and, if* $m \geqq 1$, *suppose that, almost everywhere on some set* Λ_m *in* $\mathfrak{F}_m$,