ON THE DISTRIBUTION OF QUADRATIC FORMS IN NORMAL VARIATES

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1. Introduction

If x_i is $N(0, 1)$, and the x_i are independently distributed, the function $Q_k = \sum_{i=1}^{k} a_i x_i^2$ where $a_i > 0$, arises in many situations where knowledge of its distribution is required to resolve a problem. This is true also when k is infinite and appropriate restrictions are placed on the a_i. The quadratic form which arises in the problems we will examine, and usually all others stemming from similar contexts, can always be reduced to Q_k or some known function of Q_k. Some writers have analyzed directly the distribution of Q_k as a methodological piece of work and others have investigated the distribution because it directly solved some applied problems. Among the former are papers by Robbins [8], Robbins and Pitman [9], and Hotelling [7]. In the second category there are several papers. There is a paper by von Neumann et al. [12] on the distribution of the mean square successive difference when it is used as a suitable estimator of variability when a secular trend in the mean is suspected. Grad and Solomon [5] prepared a paper on the subject which resulted from the study of a generalized hit probability problem in operations research and briefly discussed other applications. The development and the application of the distribution of a quadratic form to problems in spectral analysis especially arising from the power distribution of noise was given by Grenander, Pollak, and Slepian [6].

There are other applications. For example, in tests for goodness-of-fit when the parameters are estimated by maximum likelihood from the original observations rather than from cell frequencies and the regular chi-square statistic form is used to test the hypothesis, Chernoff and Lehmann [3] demonstrated that the distribution is the same as the distribution of Q_k. Watson [13], [14] has followed this up in subsequent papers. Billingsley [2] and Goodman [4] have demonstrated that the distribution of Q_k arises in the consideration of the asymptotic distribution of goodness-of-fit tests for stochastic processes. Anderson and Darling [1] showed that the limiting distribution of $n\omega^2$ is the distribution of $Q_\infty = \sum_{i=1}^{\infty} a_i x_i^2$ where $a_i = 1/i^2\pi^2$, and ω^2 is the von Mises criterion for goodness-of-fit between a sample cumulative distribution function and a specified population distribution function. In Rosenblatt [10], it is shown that a simple

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