STOCHASTIC APPROXIMATION

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1. Introduction

The purpose of this paper is to review the development of the so-called Robbins-Monro (RM) process. Moreover, some of the results presented here seem to be new. A summary of some results in stochastic approximation, including papers up to 1956, has been given by C. Derman [1].

The idea of stochastic approximation had its origin in the framework of sequential design (H. Robbins [2]). There are not only important applications in fields like biology, metallurgy, and so on, but also it is becoming increasingly clear that stochastic approximation is related to interesting questions in other fields of mathematics.

Let us first recall the well-known classical approach to the iterative solution of an equation of the simplest type. Suppose that M is a mapping from Euclidean space R_1 into R_1 and let α be a real number. We are interested in solutions of the equation

$$M(x) = \alpha. \tag{1}$$

It is well known that under weak assumptions on M the following is true. Let x_1 be any real number. Let us define a sequence x_n by induction,

$$x_{n+1} = x_n + a_n[\alpha - M(x_n)], \qquad n \geqq 1, \tag{2}$$

where a_n is a given sequence of real numbers which has to satisfy some conditions not enumerated here. Then x_n converges to a solution of (1); moreover, this solution is the one with the smallest distance from x_1 (R. von Mises and H. Pollaczek-Geiringer [3]).

In many practical applications it happens that the function M is only empirically given; that is to say, for every real number x the value of the function $M(x)$ is subject to an error. We suppose that for every x this error can be represented by a random variable $y(x)$ with distribution function F_x in such a way that $M(x)$ is the mathematical expectation of $y(x)$ for every real number x, so that M can be considered as a regression line. The problem is again to find a solution of equation (1). Whether one knows the error law given by F_x or not, the procedure given by (2) does not work, because we have made the assumption that $M(x)$ cannot be determined exactly. What we really can obtain is a realization for every real x of the random variable $y(x)$ whose mathematical expectation is $M(x)$. Under these circumstances one could try to define, instead