ON MEASURES OF ENTROPY AND INFORMATION

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1. Characterization of Shannon's measure of entropy

Let $\mathcal{P} = (p_1, p_2, \cdots, p_n)$ be a finite discrete probability distribution, that is, suppose $p_k \geqq 0 (k = 1, 2, \cdots, n)$ and $\sum_{k=1}^{n} p_k = 1$. The amount of uncertainty of the distribution $\mathcal{P}$, that is, the amount of uncertainty concerning the outcome of an experiment, the possible results of which have the probabilities $p_1, p_2, \cdots, p_n$, is called the *entropy* of the distribution $\mathcal{P}$ and is usually measured by the quantity $H[\mathcal{P}] = H(p_1, p_2, \cdots, p_n)$, introduced by Shannon [1] and defined by

$$H(p_1, p_2, \cdots, p_n) = \sum_{k=1}^{n} p_k \log_2 \frac{1}{p_k}. \tag{1.1}$$

Different sets of postulates have been given, which characterize the quantity (1.1). The simplest such set of postulates is that given by Fadeev [2] (see also Feinstein [3]). Fadeev's postulates are as follows.

(a) *$H(p_1, p_2, \cdots, p_n)$ is a symmetric function of its variables for $n = 2, 3, \cdots$.*

(b) *$H(p, 1 - p)$ is a continuous function of p for $0 \leqq p \leqq 1$.*

(c) *$H(1/2, 1/2) = 1$.*

(d) *$H[tp_1, (1 - t)p_1, p_2, \cdots, p_n] = H(p_1, p_2, \cdots, p_n) + p_1 H(t, 1 - t)$ for any distribution $\mathcal{P} = (p_1, p_2, \cdots, p_n)$ and for $0 \leqq t \leqq 1$.*

The proof that the postulates (a), (b), (c), and (d) characterize the quantity (1.1) uniquely is easy except for the following lemma, whose proofs up to now are rather intricate.

LEMMA. *Let $f(n)$ be an additive number-theoretical function, that is, let $f(n)$ be defined for $n = 1, 2, \cdots$ and suppose*

$$f(nm) = f(n) + f(m), \qquad n, m = 1, 2, \cdots. \tag{1.2}$$

Let us suppose further that

$$\lim_{n \to +\infty} [f(n + 1) - f(n)] = 0. \tag{1.3}$$

Then we have

$$f(n) = c \log n, \tag{1.4}$$

where c is a constant.