A MARTINGALE SYSTEM THEOREM AND APPLICATIONS

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1. Introduction

Let $(W, \mathfrak{F}, P)$ be a probability space with points $\omega \in W$ and let $(y_n, \mathfrak{F}_n)$, $n = 1, 2, \cdots$, be an *integrable stochastic sequence:* y_n is a sequence of random variables, $\mathfrak{F}_n$ is a sequence of σ-algebras with $\mathfrak{F}_n \subset \mathfrak{F}_{n+1} \subset \mathfrak{F}$, y_n is measurable with respect to $\mathfrak{F}_n$, and $E(y_n)$ exists, $-\infty \leqq E(y_n) \leqq \infty$. A random variable $s = s(\omega)$ with positive integer values is a *sampling variable* if $\{s \leqq n\} \in \mathfrak{F}_n$ and $\{s < \infty\} = W$. (We denote by $\{\cdots\}$ the set of all ω satisfying the relation in braces, and understand equalities and inequalities to hold up to sets of P-measure 0.) We shall be concerned with the problem of finding, if it exists, a sampling variable s which maximizes $E(y_s)$.

To define a sampling variable s amounts to specifying a sequence of sets $B_n \in \mathfrak{F}_n$ such that

$$0 = B_0 \subset \cdots \subset B_n \subset B_{n+1} \subset \cdots ; \bigcup_1^\infty B_n = W, \tag{1}$$

the sampling variable s being defined by

$$\{s \leqq n\} = B_n, \qquad \{s = n\} = B_n - B_{n-1}. \tag{2}$$

We shall be particularly interested in the case in which the sequence $(y_n, \mathfrak{F}_n)$ is such that the sequence of sets

$$B_n = \{E(y_{n+1}|\mathfrak{F}_n) \leqq y_n\} \tag{3}$$

satisfies (1). We shall call this the *monotone case.* In this case a sampling variable s is defined by

$$\{s \leqq n\} = \{E(y_{n+1}|\mathfrak{F}_n) \leqq y_n\}, \tag{4}$$

and s satisfies

$$E(y_{n+1}|\mathfrak{F}_n) \begin{cases} > y_n, & s > n, \\ \leqq y_n, & s \leqq n. \end{cases} \tag{5}$$

The relations (5) will be fundamental in what follows.

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