OPTIMAL GAMBLING SYSTEMS FOR FAVORABLE GAMES

L. BREIMAN
UNIVERSITY OF CALIFORNIA, LOS ANGELES

1. Introduction

Assume that we are hardened and unscrupulous types with an infinitely wealthy friend. We induce him to match any bet we wish to make on the event that a coin biased in our favor will turn up heads. That is, at every toss we have probability $p > 1/2$ of doubling the amount of our bet. If we are clever, as well as unscrupulous, we soon begin to worry about how much of our available fortune to bet at every toss. Betting everything we have on heads on every toss will lead to almost certain bankruptcy. On the other hand, if we bet a small, but fixed, fraction (we assume throughout that money is infinitely divisible) of our available fortune at every toss, then the law of large numbers informs us that our fortune converges almost surely to plus infinity. What to do?

More generally, let X be a random variable taking values in the set $I = \{1, \cdots, s\}$ such that $P\{X = i\} = p_i$ and let there be a class $\mathcal{C}$ of subsets A_j of I, where $\mathcal{C} = \{A_1, \cdots, A_r\}$, with $\bigcup_j A_j = I$, together with positive numbers $(o_1, \cdots, o_r)$. We play this game by betting amounts $\beta_1, \cdots, \beta_r$ on the events $\{X \in A_j\}$ and if the event $\{X = i\}$ is realized, we receive back the amount $\sum_{i \in A_j} \beta_j o_j$ where the sum is over all j such that $i \in A_j$. We may assume that our entire fortune is distributed at every play over the betting sets $\mathcal{C}$, because the possibility of holding part of our fortune in reserve is realized by taking A_1, say, such that $A_1 = I$, and $o_1 = 1$. Let S_n be the fortune after n plays; we say that the game is *favorable* if there is a gambling strategy such that almost surely $S_n \to \infty$. We give in the next section a simple necessary and sufficient condition for a game to be favorable.

How much to bet on the various alternatives in a sequence of independent repetitions of a favorable game depends, of course, on what our goal utility is. There are two criterions, among the many possibilities, that seem pre-eminently reasonable. One is the minimal time requirement, that is, we fix an amount x we wish to win and inquire after that gambling strategy which will minimize the expected number of trials needed to win or exceed x. The other is a magnitude condition; we fix at n the number of trials we are going to play and examine the size of our fortune after the n plays.

This research was supported in part by the Office of Naval Research under Contract Nonr-222(53).