EXPONENTIAL ERROR BOUNDS FOR FINITE STATE CHANNELS

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1. Introduction and summary

A *finite state channel* is defined by (1) a finite nonempty set A, the set of *inputs*, (2) a finite nonempty set B, the set of *outputs*, (3) a finite nonempty set T, the set of (channel) states, (4) a *transition law* $p = p(t'|t, a)$, specifying the probability that, if the channel is in state t and is given input a, the resulting state is t', and (5) a function ψ from T to B, specifying the output $b = \psi(t)$ of the channel when it is in state t.

For any sequence $\{a_n, n = 1, 2, \cdots\}$ of random variables with values in A, we may consider the process $\{a_n\}$ as supplying the inputs for the channel, as follows: an initial channel state t_0 is selected with a uniform distribution over T. The input a_1 is then given the channel. The channel then selects a state t_1, with

$$P\{t_1 = t|t_0, a_1\} = p(t|t_0, a_1) \tag{1}$$

and produces output $b_1 = \psi(t_1)$. The channel is then given input a_2 and selects state t_2, with

$$P\{t_2 = t|t_0, t_1, a_1, a_2, b_1\} = p(t|t_1, a_2), \tag{2}$$

and so on. In general, for $n \geqq 0$,

$$\begin{aligned} P\{a_{n+1} = a, t_{n+1} = t, b_{n+1} = b|a_i, 1 \leqq i \leqq n, t_i, 0 \leqq i \leqq n, b_i, 1 \leqq i \leqq n\} \\ = P\{a_{n+1} = a|a_i, i \leqq n\}p(t|t_n, a)\chi(t, b), \end{aligned} \tag{3}$$

where $\chi(t, b) = 1$ if $\psi(t) = b$ and 0 otherwise.

For any random variable x with a finite set of values and any random variable y, the (nonnegative) random variable whose value when $x = x_0$ and $y = y_0$ is

$$-\log P\{x = x_0|y = y_0\} \tag{4}$$

(all logs are base 2) is called the (conditional) entropy of x given y and will be denoted by $i(x|y)$. Its expected value, which cannot exceed the log of the number of values of x, will be denoted by $I(x|y)$. For y a constant, $i(x|y)$ and $I(x|y)$ will be denoted by $i(x)$, $I(x)$ respectively. If each of x, y has only finitely many values, the random variable

$$j(x, y) = i(x) + i(y) - i(x, y) = i(x) - i(x|y) = i(y) - i(y|x) \tag{5}$$

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