REDUCTION OF CONSTRAINED MAXIMA TO SADDLE-POINT PROBLEMS

KENNETH J. ARROW AND LEONID HURWICZ
STANFORD UNIVERSITY, UNIVERSITY OF MINNESOTA

1. Introduction

1.1. The usual applications of the method of Lagrangian multipliers, used in locating constrained extrema (say maxima), involve the setting up of the *Lagrangian expression*,

$$(1) \qquad \phi(x, y) = f(x) + y'g(x) ,$$

where $f(x)$ is being (say) maximized with respect to the (vector) variable $x = \{x_1, \cdots, x_N\}$, subject to the constraint $g(x) = 0$, where $g(x)$ maps the points of the N-dimensional x-space into an M-dimensional space, and $y = \{y_1, \cdots, y_M\}$ is the Lagrange multiplier (vector). Here, $\{\quad\}$ indicates a column vector; the prime indicates transposition, so that y' is a row vector.

The essential step of the customary procedure is the solution for x, as well as y, of the pair of (vector) equations,

$$(2) \qquad \phi_x(x, y) = 0, g(x) = 0 ,$$

where $\phi_x(x, y) = \{\partial\phi(x, y)/\partial x_1, \cdots, \partial\phi(x, y)/\partial x_N\}$. Let $(\bar{x}, \bar{y})$ be the solutions of equations (2), while $\hat{x}$ maximizes $f(x)$ subject to $g(x) = 0$. Then, under suitable restrictions,

$$(3) \qquad \bar{x} = \hat{x} .$$

1.2. In [1] Kuhn and Tucker treat the related problem of maximizing $f(x)$ subject to the constraints[1] $g(x) \geqq 0$, $x \geqq 0$, where, for an arbitrary K-dimensional vector $a = \{a_1, \cdots, a_K\}$, the relation $a \geqq 0$ is here defined to mean $a_k \geqq 0$ for $k = 1, \cdots, K$. Another definition of vectorial inequalities, permitting greater generality of treatment, will be used in later sections of this paper. There we shall treat directly the class of situations where $f(x)$ is to be maximized subject to $g^{(1)}(x) \geqq 0$, $g^{(2)}(x) = 0$, $x^{[1]} \geqq 0$, $x^{[2]}$ not restricted as to sign, $x = \{x^{[1]}, x^{[2]}\}$.

Denote by C_g the set of all x satisfying the constraints $g(x) \geqq 0$, $x \geqq 0$. The two results stated below are of fundamental importance for the problem considered.

(A) (See theorem 1 [1].) Let g satisfy the following condition (called Constraint

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[1] In [1] our f and g are respectively written as g and F. The symbol in [1] for the Lagrange multiplier (our y) is u.