RANDOM SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction

In many parts of theoretical physics one has to solve a mathematical problem of the following type:

Given (a) a partial differential equation

$$(1) \qquad \mathfrak{A}(u) = 0\,,$$

and (b) an open domain D with boundary C (in an n-dimensional Euclidean space), find a function $u(P)$ satisfying (1) in D and taking given values $f(A)$ on C,

$$(2) \qquad u(P) \to f(A) \quad \text{if} \quad P \to A\,.$$

In each case one has to specify: (a) what conditions (continuity, boundedness, existence of derivatives, etc.) $u(P)$ must satisfy in order to be considered a regular solution; (b) what precise meaning the limit in (2) has (limit along given paths, limit in the mean, etc.); and (c) what class of functions $f(A)$ (continuous, integrable, etc.) is one allowed to use for the boundary conditions, in order to secure existence and uniqueness theorems.

When one looks at the physical facts which give support to this mathematical setting, one is readily convinced that, due to errors in measurements, to neglected fluctuations in the phenomena, etc., the boundary conditions cannot be expressed by only one well-determined function $f(A)$, but that a whole set of functions $f_\omega(A)$ must be considered. Furthermore, among these we cannot identify the one that will actually materialize in an experiment; in general, we are only able to say that some functions in the set are more likely to be observed than some others. We can translate this idea in mathematical language by saying that, in the boundary conditions corresponding to the real physical problem, the function $f(A)$ must be replaced by a random function $f_\omega(A)$, where ω represents a point in a suitable probability space $(\Omega, \mathfrak{F}, \mu)$. We have been led to this point of view by our research on the statistical theory of turbulence [2] and [3]. In our opinion the key problem in this theory is to find random solutions of the Navier-Stokes equations corresponding to a given random velocity field at time $t = 0$. But, unfortunately, since the equations of fluid dynamics are nonlinear we know almost nothing about existence and uniqueness of their solutions. Since we are attracted to this point of view and yet unable to solve this most interesting problem, we have had to satisfy ourselves by considering only linear partial differential equations. In this case it is possible to prove that the consideration of random boundary values makes sense and to build a rather complete theory of the corresponding random solutions.