A NOTE ON RANDOM TRIGONOMETRIC POLYNOMIALS

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1. General remarks

This note is a postscript to our paper [1]. It deals with a problem having close connection with the topics discussed there, and uses similar methods. However, to make the note more readable, we make it self-contained at the expense of a repetition of some of the arguments in [1]. For the sake of proper perspective we begin by restating some of the results of that paper.

Consider a general trigonometric polynomial of order n,

$$\frac{1}{2}a_0+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)\,, \tag{1.1}$$

with, say, real coefficients. Let $\varphi_1(t), \varphi_2(t),\cdots, \varphi_n(t),\cdots$ be the Rademacher functions

$$\varphi_n(t)=\text{sign}\sin 2^n\pi t\,, \qquad 0\leqq t\leqq 1\,, \tag{1.2}$$

which represent independent random variables taking values ± 1, each with probability 1/2. We write

$$P_n(x,t)=\frac{1}{2}a_0+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)\,\varphi_k(t)\,, \tag{1.3}$$

$$M_n(t)=\max_x|P_n(x,t)|\,. \tag{1.4}$$

One of the problems discussed in [1] was that of the order of magnitude of $M_n(t)$ for $n\rightarrow\infty$ and almost all t (this presupposes, of course, that the a_k and b_k are defined for all k). It turns out (see pp. 270–271 in [1]) that, *if the series* $\sum(a_k^2+b_k^2)$ *diverges, and*

$$R_n=\frac{1}{2}\sum_{k=1}^{n}(a_k^2+b_k^2)\,, \tag{1.5}$$

then

$$\limsup_{n\rightarrow\infty}\frac{M_n(t)}{\sqrt{R_n\log n}}\leqq 2 \tag{1.6}$$

for almost all t.

This result was obtained under the sole assumption that $\sum(a_k^2+b_k^2)$ diverges. If we want to obtain an estimate for $M_n(t)$ from below we must introduce further restrictions on a_n, b_n. Write

$$T_n=\sum_{k=1}^{n}(a_k^4+b_k^4)\,. \tag{1.7}$$