RANKING LIMIT PROBLEM

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1. The problem

Let $(\Omega, \mathcal{A}, P)$ be our probability space and let X, with or without affixes, denote a measurable function [a random variable (r.v.) when finite] on this space. $\mathcal{L}(X)$ will represent the (probability) law of X defined by its distribution function (d.f.) F or its characteristic function (ch. f.) f with the same affixes as X, if any. A law degenerate at a is represented by $\mathcal{L}(a)$; if a is finite, it is the law of a r.v. which reduces to a with probability 1; $\mathcal{L}(\infty)$ represents the law of any measurable function which is infinite with probability 1.

Distribution functions and, more generally, monotone functions, say, h on $R = (-\infty, +\infty)$, will be continuous from the left: $h(x-0) = h(x)$, $x \in R$. A sequence h_n of monotone functions, say, nondecreasing ones, converges *weakly* to h on R, and we write $h_n \xrightarrow{w} h$, if $h_n \to h$ on the continuity set of h (it suffices that $h_n \to h$ on a set everywhere dense in R); h_n converges *completely* to h, and we write $h_n \xrightarrow{c} h$, if, moreover, $h_n(\mp\infty) \to h(\mp\infty)$. A sequence of laws $\mathcal{L}(X_n)$ converges weakly or completely to a law $\mathcal{L}(X)$ if $F_n \to F$ weakly or completely, respectively.

Convention I. Throughout this paper, and unless otherwise stated,

(a) To any probability p we make correspond the probability $q = 1 - p$ with the same affixes, if any.

(b) $n = 1, 2, \cdots$; $k = 1, 2, \cdots, k_n$, with $k_n \to \infty$; all limits are taken for $n \to \infty$.

(c) X_{nk} represent r.v.'s independent in k for every fixed n. For every $\omega \in \Omega$, the nondecreasingly ranked numbers $X_{nk}(\omega)$ are denoted by

$$X_{n1}^*(\omega) \leqq X_{n2}^*(\omega) \leqq \cdots \leqq X_{nk_n}^*(\omega); \tag{1}$$

they are values of nondecreasingly *ranked r.v.'s* X_{nr}^*, $r = 1, 2, \cdots, k_n$, of *rank* r and *relative rank* $\rho = r/k_n$ (with the same affixes as r, if any), corresponding to the r.v.'s X_{nk}. The nonincreasingly ranked r.v.'s are denoted by ${}^*X_{ns}$, $s = 1, 2, \cdots, k_n$, of *end rank* s, so that ${}^*X_{ns} = X_{n,\,k_n+1-s}^*$.

Let the X_{nk} be uniformly asymptotically negligible, that is, $\mathcal{L}(X_{nk}) \to \mathcal{L}(0)$ uniformly in k. We know that *if* $\mathcal{L}\left(\sum_k X_{nk}\right) \xrightarrow{c} \mathcal{L}(X)$, *then* $\mathcal{L}(X)$ *is infinitely decomposable*. We recall that a law $\mathcal{L}(X)$ is infinitely decomposable, that is, $f^{1/n}$ is a ch. f. for every n if, and only if, for every $u \in R$

$$\log f(u) = iau - \frac{b^2}{2}u^2 + \int_{-\infty}^{-0} \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) dL(x) + \int_{+0}^{+\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) dM(x), \tag{2}$$

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