A SPECIAL PROBLEM OF BROWNIAN MOTION, AND A GENERAL THEORY OF GAUSSIAN RANDOM FUNCTIONS

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1. Introduction

1.1. The subject of this paper is twofold: a special problem and a general theory. The reader may wonder why the general theory is not stated in part 2 and then applied to the special problem. The answer is that the general theory appeared as a necessary generalization of theorems stated in part 2, after the two first parts had been written, and the author thought that he would not have enough time before this Symposium to reorganize the paper. Moreover, part 2 will be a good introduction to the general theory. In the introduction the author will begin with the general theory, and the reader who wishes to do so may begin with part 3.

1.2. The problem considered in this theory is to find an explicit representation of a Gaussian r.f.[1] of a real variable t that may be considered as the canonical form of this function. By subtraction of a known function, it may be reduced to a Gaussian r.f. $\phi(t)$ with identically zero expectation. Such a r.f. is generally defined by its covariance $\Gamma(t_1, t_2)$, or by a stochastic differential equation with a Cauchy condition. None of these methods gives an explicit representation of $\phi(t)$.

In his previous papers [5] and [6], the author has considered the relation between these two classical methods, and solved the problem of deducing one of these representations from the other. He has also called attention to an explicit representation, which may be written in the form

$$\phi(t) = \int_0^t F(t, u)\, \xi_u \sqrt{du}, \tag{1.2.1}$$

where the r.v. ξ_u are independent reduced Gaussian r.v. Here $\xi_u\sqrt{du}$ may be replaced by $dX(u)$, where $X(u)$ is the Wiener r.f. (see, for instance, formula (3.2.15) in [6]). We shall call $\mu + \sigma\xi$ the *canonical form* of a Gaussian X, where ξ is a reduced Gaussian r.v., μ is the expectation of X, and σ is its standard deviation. This leads naturally to the following definition: if the conditional canonical form of $\phi(t')$ is $\mu(t'\,|\,t) + \xi\sigma(t'\,|\,t)$ when $\phi(u)$ is given in $(0, t)$ (with $0 < t < t'$), then formula (1.2.1) gives the *canonical form* of $\phi(t)$ if

$$\mu(t'\,|\,t) = \int_0^t F(t', u)\, \xi_u \sqrt{du}. \tag{1.2.2}$$

This paper was supported (in part) by funds provided under Contract AF-18(600)-958 with the Air Research and Development Command.

[1] We shall use the following abbreviations: r.v., random variable (or variables); r.f., random function (or functions); a.s., almost sure (or surely).