# ISOTROPIC RANDOM CURRENT

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## 1. Introduction

The theory of isotropic random vector fields was originated by H. P. Robertson [1] in his theory on isotropic turbulence. He defined the covariance bilinear form of random vector fields which corresponds to Khinchin's covariance function in the theory of stationary stochastic processes. Although in the latter theory the essential point was made clear in connection with the theory of Hilbert space and that of Fourier analysis, we have no corresponding theory on isotropic random vector fields.

Robertson obtained a condition necessary for a bilinear form to be the covariance bilinear form of an isotropic random vector field. Unfortunately his condition is not sufficient; in fact, he took into account only the invariant property of the covariance bilinear form but not its positive definite property. A necessary and sufficient condition was obtained by S. Itô [2]. Although his statement is complicated, he grasped the crucial point. His result corresponds to Khinchin's spectral representation of the covariance function of stationary stochastic processes.

The purpose of this paper is to establish a general theory on homogeneous or isotropic random vector fields, or more generally the homogeneous or isotropic random currents of de Rham [3]. In section 2 we shall give a summary of some known facts on vector analysis for later use. In section 3 we shall define random currents and random measures. The reason we treat random currents rather than random $p$-vector fields or $p$-form fields is that we have no restrictions in applying differential operators $d$ and $\delta$ to random currents. These operators will elucidate the essential point. In section 4 we define homogeneous random currents and give spectral representations. Here we shall explain the relation between homogeneous random currents and random measures. In section 5 we shall show a decomposition of a homogeneous random current into its irrotational part, its solenoidal part and its invariant part. In the next section we shall give a spectral representation of the covariance functional of an isotropic random current. The result here contains S. Itô's formula as a special case. The spectral measure in this representation is decomposed into three parts which correspond to the above three parts in the decomposition of a homogeneous random current. This relation was not known to S. Itô. In [4] we have shown that the Schwartz derivative of the Wiener process is a stationary random distribution which is not itself a process. A similar fact will be seen in section 7 with respect to the gradient of P. Lévy's Brownian motion [5] with a multidimensional parameter.

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