THE ZEROS OF A RANDOM POLYNOMIAL

J. M. HAMMERSLEY
OXFORD UNIVERSITY

1. Introduction

If the coefficients of an algebraic equation are subject to random error, the roots of this equation will also be subject to random error; and it is natural to enquire how the latter errors depend upon the former. This is clearly a question of some practical importance. It arises not only when the coefficients result from experimental data, but also, for example, when the coefficients are rounded off to some specified number of decimal places before commencing a numerical solution. Yet it is a question which has so far received rather scant attention, apart from the treatment of three special instances.

In the first of these special instances the equation is of a particular type, namely the characteristic equation of a variance-covariance matrix pencil whose elements are real and distributed in Wishart's form. Under these circumstances the roots are all real, and their joint sampling distribution is well known. Amongst the several textbooks, which discuss this question, the reader may consult Wilks [19].

In the second special instance the equation is linear and the coefficients are real and distributed normally (though not necessarily independently). The distribution of the root of this equation is therefore that of the quotient of two real correlated normal variates. Geary [9] gives the required result. This special instance arises in bio-assay work under the name of Fieller's theorem (see pp. 27–29 in Finney [8]). The interpretation of this theorem is, however, open to question, and I discuss this matter further in section 9.

In the third special instance the equation is of general degree and its coefficients are real and distributed independently and symmetrically about zero either

(i) normally, or

(ii) rectangularly, or

(iii) discretely into the pair of classes ± 1.

Littlewood and Offord [14], [15] enquired how many real roots on the average such a random equation might be expected to have, and they gave asymptotic approximations for the result valid for equations of large degree. Kac [12], [13] improved their results by showing that the average number of real roots of an equation of degree $n-1$ is

$$N_n = \frac{4}{\pi}\int_0^1 \frac{(1-h_n^2)^{1/2}}{1-x^2}\,dx \leqq \frac{2}{\pi}\log n + \frac{14}{\pi}, \tag{1.1}$$

where

$$h_n = n x^{n-1}\frac{1-x^2}{1-x^{2n}} \tag{1.2}$$

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