RANDOM DISTRIBUTIONS WITH AN APPLICATION TO TELEPHONE ENGINEERING

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1. General Poisson distributions

Recently, A. Blanc-Lapierre and I remarked that we were not aware of any definition of a general Poisson distribution, that is to say, of a Poisson distribution not, as usual, on the straight line or on some Euclidean space, but on a perfectly general space. Such a definition may be useful, and can be given in the following obvious way.

Let $\mathcal{X}$ be any space of elements x, $\mathcal{B}$ a Borel field of subsets e of $\mathcal{X}$, and $m(e)$ a measure on $\mathcal{B}$ (not necessarily bounded or finite). A random family F of elements of $\mathcal{X}$ is a Poisson distribution on $\mathcal{X}$ [with respect to $\mathcal{B}$ and $m(e)$] if, $M(e)$ being the number of elements of F belonging to $e \in \mathcal{B}$, we have the following properties [1]:

1) If $m(e) < +\infty$, the random variable $M(e)$ is almost certainly finite and its distribution function is the Poisson law with parameter $m(e)$.

2) If k is any integer and if $e_1, e_2, \cdots, e_k$ are any disjoint sets belonging to $\mathcal{B}$, with $m(e_j) < +\infty, j = 1, 2, \cdots, k$, the k random variables $M(e_j)$ are independent.

The classical properties of Poisson distributions on the straight line remain true. For instance, it is easy to see the following.

1) If $e \in \mathcal{B}$ with $0 < m(e) < +\infty$, then conditionally when $M(e) = k$, with k any integer >0, the distribution on e of the k elements of F belonging to e is statistically equivalent to the choice at random, independently, of k elements x on e, with $Pr\{x \in e'\} = m(e')/m(e)$, where e' is any subset of e belonging to $\mathcal{B}$.

2) Let e_t be a family of sets belonging to $\mathcal{B}$, $0 \leqq t < +\infty$, such that (a) $e_t \subset e_\tau$ if $t < \tau$; (b) e_0 reduces to an element $x_0 \in \mathcal{X}$; (c) $m(e_t)$, as a function of t, is continuous even for $t \rightarrow +0$ with $m(e_0) = 0$. Let T be the random variable defined by the following. If $t < T$, no element of F belongs to e_t; if $t > T$, at least one element of F belongs to e_t. The distribution function of T is

$$(1.1) \qquad 1 - e^{-m(e_t)} .$$

3) Let $\mathfrak{Y}$ be a second space, $\mathcal{C}$ a Borel field of subsets $\omega \subset \mathfrak{Y}$, let $p(x; \omega)$ be a probability measure on $\mathcal{C}$ corresponding to every $x \in \mathcal{X}$, and let $Y(x)$ be a random element taking its values on $\mathfrak{Y}$ obeying the law $p(x; \omega)$. I assume that the different $Y(x)$ for different x are mutually independent. Let $e \in \mathcal{B}$ with $m(e) < +\infty$, let $X_1, X_2, \cdots, X_j, \cdots$ be the elements of F belonging to e. Then the $Y(X_j)$ are Poisson distributed on $\mathfrak{Y}$ (with respect to $\mathcal{C}$) and the mathematical expectation of the number of the $Y(X_j)$ belonging to ω is

$$(1.2) \qquad \int_e p(x; \omega)\, m(dx) .$$

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