PROBABILITY METHODS APPLIED TO THE FIRST BOUNDARY VALUE PROBLEM

J. L. DOOB
UNIVERSITY OF ILLINOIS

1. Introduction

The first boundary value problem (sometimes called the Dirichlet problem) is, in its most restricted form, that of finding a function u, defined on a specified open set D, such that u is a member of some specified linear class, whose members we shall call *regular* functions, and that u has a prescribed continuous boundary function. In the more general form of the problem, the hypothesis of continuity of the boundary function is dropped, and the connection between the function u and the boundary function f is correspondingly loosened. Even the restricted problem, however, cannot usually be solved in its original form without either imposing restrictions on the boundary of D or loosening the relation between u and f.

In many classical cases, $u(z)$, the value of the solution at z, becomes a positive linear functional of the continuous boundary function f, for each point z of D, with value 1 when f is the constant function 1. In view of the Riesz representation theorem, or of the Daniell approach to integration if the Riesz theorem is not available, $u(z)$ can then be expressed as a weighted average of f. In this paper, we shall treat the first boundary value problem for functions defined in the first place by the property that they are, in specified domains, weighted averages of their values over the boundaries. The results will have wide applicability because very few conditions are imposed on the underlying space, the specified domains, or the averaging method. The Perron-Wiener-Brelot (PWB) method is applied to solve a generalized version of the first boundary value problem. Not enough hypotheses are imposed, however, to make the method lead to a solution for all continuous bounded boundary functions. That is, in Brelot's terminology, not all such boundary functions need be resolutive. The PWB method is then slightly generalized, using probability methods, to obtain a slightly larger and more manageable class of resolutive boundary functions, and thus to put in a more general setting the fact that the original PWB solutions are simple weighted averages of the specified boundary functions.

The preceding results, given the domain and class of functions under consideration, are substantially independent of the way in which the domain boundary is defined, even though a PWB solution is shown to have the specified boundary function as a limit along appropriate paths to the boundary. This fact is put in a more general setting by defining and solving a new type of first boundary value problem, an intrinsic problem, in which there is no specified boundary, but in which the general properties of regular functions are used to define a specified limit behavior of a regular function outside compact sub-

This paper was prepared with the support of the Office of Ordnance Research, U.S. Army, under Contract DA-04-200-ORD-355.