ON A CLASS OF PROBABILITY SPACES

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1. Introduction

Kolmogorov's model for probability theory [10], in which the basic concept is that of a probability measure P on a Borel field $\mathcal{B}$ of subsets of a space Ω, is by now almost universally considered by workers in probability and statistics to be the appropriate one. In 1948, however, three somewhat disturbing examples were published by Dieudonné [2], Andersen and Jessen [1], and Doob [3] and Jessen [9], as follows.

A. (Dieudonné). There exist a pair $(\Omega, \mathcal{B})$, a probability measure P on $\mathcal{B}$, and a Borel subfield $\mathcal{A} \subset \mathcal{B}$ for which there is no function $Q(\omega, E)$ defined for all $\omega \in \Omega$, $E \in \mathcal{B}$ with the following properties: Q is for fixed E an $\mathcal{A}$-measurable function of ω, for fixed ω a probability measure on $\mathcal{B}$, and for every $A \in \mathcal{A}$, $E \in \mathcal{B}$, we have

$$\int_A Q(\omega, E)\, dP(\omega) = P(A \cap E). \tag{1}$$

B. (Andersen and Jessen). There exist a sequence of pairs $(\Omega_n, \mathcal{B}_n)$ and a function P defined for all sets of $\cup \mathcal{A}_n$, where $\mathcal{A}_n$ consists of all subsets of the infinite product space $\Omega_1 \times \Omega_2 \times \cdots$ in the Borel field determined by sets of the form $B_1 \times \cdots \times B_n \times \Omega_{n+1} \times \Omega_{n+2} \times \cdots$, $B_i \in \mathcal{B}_i$, $i = 1, \cdots, n$, such that P is countably additive on each $\mathcal{A}_n$ but not on $\cup \mathcal{A}_n$.

C. (Doob, Jessen). There exist a pair $(\Omega, \mathcal{B})$, a probability measure P on $\mathcal{B}$, and two real-valued $\mathcal{B}$-measurable functions f, g on Ω such that

$$P\{\omega\colon f \in F,\ g \in G\} = P\{\omega\colon f \in F\} P\{\omega\colon g \in G\} \tag{2}$$

holds for every two linear Borel sets F, G but not for every two linear sets F, G for which the three probabilities in (2) are defined.

In each case Ω is the unit interval, $\mathcal{B}$ is the Borel field determined by the Borel sets and one or more sets of outer Lebesgue measure 1 and inner Lebesgue measure 0, and P consists of a suitable extension of Lebesgue measure to $\mathcal{B}$. The fact that A, B, C cannot happen if Ω is a Borel set in a Euclidean space and $\mathcal{B}$ consists of the Borel subsets of Ω is known. For A, the proof was given by Doob [4], for B by Kolmogorov [10], and for C by Hartman [7].

To the extent that A, B, C violate one's intuitive concept of probability, they suggest that the Kolmogorov model is too general, and that a more restricted concept, in which A, B, C cannot happen, is worth considering. In their book [5], Gnedenko and Kolmogorov propose a more restricted concept, that of a *perfect* probability space, which is a triple $(\Omega, \mathcal{B}, P)$ such that for any real-valued $\mathcal{B}$-measurable function f and any linear set A for which $\{\omega\colon f(\omega) \in A\} \in \mathcal{B}$, there is a Borel set $B \subset A$ such that

$$P\{\omega\colon f(\omega) \in B\} = P\{\omega\colon f(\omega) \in A\}. \tag{3}$$

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