AN EMPIRICAL BAYES APPROACH TO STATISTICS

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Let X be a random variable which for simplicity we shall assume to have discrete values x and which has a probability distribution depending in a known way on an unknown real parameter Λ,

$$p(x \mid \lambda) = Pr\,[X = x \mid \Lambda = \lambda]\,, \tag{1}$$

Λ *itself being a random variable* with *a priori* distribution function

$$G(\lambda) = Pr\,[\Lambda \leqq \lambda]\,. \tag{2}$$

The unconditional probability distribution of X is then given by

$$p_G(x) = Pr\,[X = x] = \int p(x \mid \lambda)\, dG(\lambda)\,, \tag{3}$$

and the expected squared deviation of any estimator of Λ of the form $\varphi(X)$ is

$$\begin{aligned} E[\varphi(X) - \Lambda]^2 &= E\{E[(\varphi(X) - \Lambda)^2 \mid \Lambda = \lambda]\} \\ &= \int \sum_x p(x \mid \lambda)[\varphi(x) - \lambda]^2 dG(\lambda) \\ &= \sum_x \int p(x \mid \lambda)[\varphi(x) - \lambda]^2 dG(\lambda)\,, \end{aligned} \tag{4}$$

which is a minimum when $\varphi(x)$ is defined for each x as that value $y = y(x)$ for which

$$I(x) = \int p(x \mid \lambda)(y - \lambda)^2 dG(\lambda) = \text{minimum}\,. \tag{5}$$

But for any fixed x the quantity

$$\begin{aligned} I(x) &= y^2 \int p\,dG - 2y \int p\lambda\,dG + \int p\lambda^2 dG \\ &= \int p\,dG \left(y - \frac{\int p\lambda\,dG}{\int p\,dG}\right)^2 + \left[\int p\lambda^2 dG - \frac{(\int p\lambda\,dG)^2}{\int p\,dG}\right] \end{aligned} \tag{6}$$

is a minimum with respect to y when

$$y = \frac{\int p\lambda\,dG}{\int p\,dG}\,, \tag{7}$$

the minimum value of $I(x)$ being

$$I_G(x) = \int p(x \mid \lambda)\,\lambda^2 dG(\lambda) - \frac{[\int p(x \mid \lambda)\,\lambda\,dG(\lambda)]^2}{\int p(x \mid \lambda)\,dG(\lambda)}\,. \tag{8}$$

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