NONLINEAR PROGRAMMING

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1. Introduction

Linear programming deals with problems such as (see [4], [5]): to maximize a linear function $g(x) \equiv \sum c_i x_i$ of n real variables $x_1, \ldots, x_n$ (forming a vector x) constrained by $m + n$ linear *inequalities*,

$$f_h(x) \equiv b_h - \sum a_{hi} x_i \geqq 0, \; x_i \geqq 0, \qquad h = 1, \ldots, m; \; i = 1, \ldots, n.$$

This problem can be transformed as follows into an equivalent saddle value (minimax) problem by an adaptation of the calculus method customarily applied to constraining *equations* [3, pp. 199–201]. Form the Lagrangian function

$$\phi(x, u) \equiv g(x) + \sum u_h f_h(x).$$

Then, a particular vector x^0 maximizes $g(x)$ subject to the $m + n$ constraints if, and only if, there is some vector u^0 with nonnegative components such that

$$\phi(x, u^0) \leqq \phi(x^0, u^0) \leqq \phi(x^0, u) \quad \text{for all nonnegative } x, u.$$

Such a saddle point (x^0, u^0) provides a solution for a related zero sum two person game [8], [9], [12]. The bilinear symmetry of $\phi(x, u)$ in x and u yields the characteristic duality of linear programming (see section 5, below).

This paper formulates necessary and sufficient conditions for a saddle value of any differentiable function $\phi(x, u)$ of nonnegative arguments (in section 2) and applies them, through a Lagrangian $\phi(x, u)$, to a maximum for a differentiable function $g(x)$ constrained by inequalities involving differentiable functions $f_h(x)$ mildly qualified (in section 3). Then, it is shown (in section 4) that the above equivalence between an inequality constrained maximum for $g(x)$ and a saddle value for the Lagrangian $\phi(x, u)$ holds when $g(x)$ and the $f_h(x)$ are merely required to be concave (differentiable) functions for nonnegative x. (A function is *concave* if linear interpolation between its values at any two points of definition yields a value not greater than its actual value at the point of interpolation; such a function is the negative of a *convex* function—which would appear in a corresponding minimum problem.) For example, $g(x)$ and the $f_h(x)$ can be quadratic polynomials in which the pure quadratic terms are negative semidefinite (as described in section 5).

In terms of *activity analysis* [11], x can be interpreted as an activity vector, $g(x)$ as the resulting output of a desired commodity, and the $f_h(x)$ as unused balances of primary commodities. Then the Lagrange multipliers u can be interpreted as a price vector [13, chap. 8] corresponding to a unit price for the desired commodity, and the Lagrangian function $\phi(x, u)$ as the combined worth of the output of the desired commodity and the unused balances of the primary commodities. These

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