A REMARK ON CHARACTERISTIC FUNCTIONS

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Let $F_1(t), F_2(t), \ldots, F_n(t), \ldots$ be a sequence of distribution functions, and let

$$\varphi_n(x) = \int_{-\infty}^{+\infty} e^{ixt} dF_n(t)$$

be the corresponding characteristic functions. If the sequence $\{\varphi_n(x)\}$ converges over every finite interval, and if the limit is continuous at the point $x = 0$, then, as is very well known, the sequence $\{F_n(t)\}$ converges to a distribution function $F(t)$ at every point of continuity of the latter (see, for example, [1, p. 96]. It is also very well known that in this theorem convergence over every finite interval cannot be replaced by convergence over a fixed interval containing the point $x = 0$.

The situation is different if the random variables whose distribution functions are the F_n are uniformly bounded below (or above). Without loss of generality we may assume that the random variables in question are positive, so that all $F_n(t)$ are zero for t negative. The purpose of this note is to prove the following theorem.

THEOREM. *Let $F_1(t), F_2(t), \ldots, F_n(t), \ldots$ be a sequence of distribution functions all vanishing for $t \leqq 0$, and let*

$$\varphi_n(x) = \int_0^{+\infty} e^{ixt} dF_n(t), \qquad -\infty < x < +\infty .$$

If the functions $\varphi_n(x)$ tend to a limit in an interval around $x = 0$, and if the limiting function is continuous at $x = 0$, then there is a distribution function $F(t)$ such that $F_n(t)$ tends to $F(t)$ at every point of continuity of F.

PROOF. Let $z = x + iy$, and let us consider the functions

$$\varphi_n(z) = \int_0^{+\infty} e^{izt} dF_n(t) = \int_0^{+\infty} e^{ixt} e^{-yt} dF_n(t) .$$

Each $\varphi_n(z)$ is regular for $y > 0$, continuous for $y \geqq 0$, and is of modulus $\leqq 1$ there. For z real, $\varphi_n(z)$ coincides with the characteristic function $\varphi_n(x)$. It is easy to see that the sequence $\{\varphi_n(z)\}$ converges in the half plane $y > 0$, and that the convergence is uniform over any closed and bounded set of this half plane. For let $z = \lambda(\zeta)$ be a conformal mapping of the half plane $y > 0$ onto the unit circle $|\zeta| < 1$, and let us consider the functions

$$\varphi_n^*(\zeta) = \varphi_n[\lambda(\zeta)] . \tag{1}$$

These functions are regular for $|\zeta| < 1$, are numerically $\leqq 1$ there and their