SOME PROBLEMS ON RANDOM WALK IN SPACE

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1. Introduction

Consider the lattice formed by all points whose coordinates are integers in d-dimensional Euclidean space, and let a point $S_d(n)$ perform a move randomly on this lattice according to the following rules: At time zero it is at the origin and if at any time $n - 1$ $(n = 1, 2, \ldots)$ it is at some point S then at time t it will be at one of the $2d$ lattice points nearest S, the probability of it being at any specified one of those being $1/(2d)$. In 1921 G. Pólya [7] discovered the remarkable fact that a point moving randomly according to the rules explained above will, with probability 1, return infinitely often to the origin if $d \leqq 2$ while if $d > 2$ then it will, again with probability 1, wander off to infinity.

While the random walk on the line has been very extensively studied there were relatively few studies of random walk in the plane or in the space. In particular many problems arising in connection with the above mentioned results of Pólya have been completely neglected. This is somewhat unfortunate since these questions, besides being of intrinsic interest, also arise in certain physical and statistical investigations. In the present paper we study two asymptotic problems concerning random walk.

The first problem is concerned with the number of different lattice points through which the random walk path passes; it is studied in sections 2–5. The other problem is that of the rate with which a point walking randomly in d-space $(d \geqq 3)$ escapes to infinity and is studied in section 6. The treatments of the two problems are independent.

We find that during the first n steps a random path in the plane passes in the average through approximately $\pi n/\log n$ different points, while in d-space $(d \geqq 3)$ it passes through approximately $n\gamma_d$ different points (with $0 < \gamma_d < 1$). We also estimate the variance of the number of different points covered and show that not only weak but even strong laws of large numbers hold. The proof of the strong law in the plane (section 5) is considerably more difficult than in d-space for $d \geqq 3$ (section 4). We deliberately refrain from applying our methods to the same problem in one dimensional random walk (for some remarks on this problem see Erdös [2]).

In section 6 we characterize all monotone functions $g(n)$ having the property that, with probability 1,[1]

$$g(n)\sqrt{n} = o\,[\|S_d(n)\|] \qquad d = 3, 4, \ldots \tag{1.1}$$

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[1] o and O always refer to the relevant variable (usually n) tending to infinity.