

# THE STRONG LAW OF LARGE NUMBERS

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## 1. Introduction

A well known unsolved problem in the theory of probability is to find a set of necessary and sufficient conditions (nasc's) for the validity of the strong law of large numbers (SLLN) for a sequence of independent random variables. This problem will not be solved in the present paper. To avoid a possible misunderstanding it must be stated at once that nasc's have been found, and several sets of them will be given in section 3, but they are all unsatisfactory. Presumably all (or shall we say most) mathematicians will agree on a satisfactory set of such conditions if and when they are exhibited, but before they are it does not seem easy to lay down criteria of satisfactoriness. On the other hand it is safe to rule out certain conditions as unsatisfactory, for example those in which sums of random variables enter; the conditions to be given in section 3 all have this undesirable property.

The purpose of this paper is to give an account of the latest information on this problem, at least in some directions. While undoubtedly much that follows is known to experts in the field or, so to speak, lurks in the corners of their minds, it is hoped that some of the results below are printed here for the first time and not sufficiently known to a wider circle of probabilists. It is to acquaint this latter group with the present status of knowledge of the problem that this paper is written.

The paper is divided into three sections. Section 2 is quite independent of the others and deals with the case of identically distributed, independent random variables (r.v.'s). In this case it is known, after Kolmogorov,<sup>1</sup> that a nasc for the validity of the SLLN is the finiteness of the first absolute moment of the common distribution function (d.f.). For use in certain statistical applications Professor Wald raised the question of the uniformity of the strong convergence with respect to a family of d.f.'s (see section 2). A nasc for this is given in section 2, which includes Kolmogorov's theorem as a special case. The method of proof is classical.

In section 3 several sets of nasc, but unsatisfactory, conditions for the validity of the SLLN are given and their interrelations, mostly trivial, are explored. The results of this section includes Kawata's partial result<sup>2</sup> in this direction, and Pro-

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<sup>1</sup> See Kolmogorov [1, p. 67]. As far as the author is aware the proof was never published by him. The proof of the sufficiency part is given in Fréchet [2]. The necessity part has been given without centering at the medians; see Feller [3], for more general results.

<sup>2</sup> Kawata [4]. He stated the theorem with zero expectations, an assumption which he never used.