# ON ALMOST SURE CONVERGENCE

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## 1. Introduction

Since the discovery by Borel[1] (1907) of the strong law of large numbers in the Bernoulli case, there has been much investigation of the problem of almost sure convergence and almost sure summability of series of random variables. So far most of the results concern series of independent random variables. In the case of dependent random variables, the first general result is the celebrated Birkhoff ergodic theorem [1], or the strong law of large numbers for a stationary sequence with a finite first moment. This theorem contains the Kolmogoroff strong law of large numbers as a particular case. P. Lévy studied series that are the same as those of independent random variables in their properties of second order as described by the first and second conditional moments. The author [8] investigated series which behave asymptotically as those of P. Lévy. There are also properties of martingales, due essentially to Doob [2], P. Lévy and Ville [10].

We shall proceed to a systematic investigation of almost sure convergence of sequences of random variables, emphasizing the methods and assuming as little as possible about the stochastic structure of the sequences. The known results will appear as various particular cases of a few propositions, and the necessary known tools will be established. In that respect, this paper is self contained.

Part I is devoted to definitions, notations, and general criteria. In part II the truncation and centering ideas are expounded. Part III is concerned with the use of "determining" set functions and with two propositions. The particular cases of one of them contain the martingale properties and of the other the ergodic theorem and its known extensions.

## PART I. BASIC CONCEPTS AND ELEMENTARY INEQUALITIES

## 2. Fundamental notions

Let $(F, P)$ represent a *probability field* defined on a space $\Omega$ of "points" $\omega$.

$F$ is a $\sigma$-field of *events* $A$ in $\Omega$, that is a family of sets in $\Omega$ such that (i) the *sure event* $\Omega$ belongs to $F$, (ii) if $A \in F$, then $A' = (\Omega - A) \in F$, (iii) if $A_n \in F$ for $n = 1, 2, \ldots$, then $\bigcap_n A_n \in F$. It follows then, from $(\bigcup_n A_n)' = \bigcap_n A_n'$, that (iii)$'$ if $A_n \in F$ for $n = 1, 2, \ldots$, then $\bigcup_n A_n \in F$ and, in general, if a sequence of events has a limit, the limit is also an event. When the events are disjoint we shall repre-

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[1] References, denoted by numbers in square brackets, are listed at the end of the paper except for those of the known results which can be found in P. Lévy, *Théorie de l'addition des variables aléatoires*, Gauthier-Villars (1937).