CONTINUOUS PARAMETER MARTINGALES

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1. Introduction

Let Ω be an abstract space of points ω, and let $Pr\{\cdot\}$ be a probability measure defined on some Borel field of ω sets. The sets of this field will be called *measurable.* A family of (real or complex valued) random variables, that is of measurable ω functions, is called a *stochastic process.* We shall use the notation $\{x(t),\ t \in T\}$ to denote a stochastic process. Here T is the parameter set of the process, and $x(t)$ is for each $t \in T$ a random variable, taking on the value $x(t, \omega)$ for given t, ω. For fixed ω, $x(t, \omega)$ determines a function $x(\cdot, \omega)$ of $t \in T$. The functions of t determined in this way are called the *sample functions* (or sample sequences if T is finite or denumerable) of the process. The random variable $x(t)$ can also appropriately be denoted by $x(t, \cdot)$, but the latter notation will not be used. The phrase *almost all sample functions* will mean *for almost all* ω.

Suppose that our old friend Peter is playing a fair game with his old friend Paul (or suppose that the classical situation is modernized, so that a SCIENTIST plays NATURE). Suppose that at time t our protagonist has fortune $x(t)$. One mathematical version of a fair game is obtained by supposing that $x(t)$ is a random variable, and that our protagonist's expected fortune at time t, in view of his previous fortunes up to time $s < t$, is simply $x(s)$. More precisely our mathematical version of a fair game is a stochastic process $\{x(t),\ t \in T\}$ for which T is a simply ordered set, for which

$$E\{|x(t)|\} < \infty, \qquad t \in T,$$

and for which

$$E\{x(t) | x(r),\quad r \leqq s\} = x(s)$$

with probability 1, if $s < t$. A stochastic process satisfying these conditions is called a *martingale.*

If x is a random variable, it will be convenient to denote the ω set where $x(\omega) \in A$ by $\{x \in A\}$. Here and in the following, in this connection, it will be understood that A is a linear set if the random variable x is real and a plane set if x is complex. The ω measure of the indicated set will be denoted by $Pr\{x \in A\}$, if this ω set is measurable. The corresponding conventions are made if more than one random variable is involved. The integral of a random variable x on a measurable set Λ will be denoted by

$$\int_\Lambda x\,dPr\,.$$

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