A PROBLEM ON RANDOM WALK

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1. From an urn containing an equal number of each of l kinds of balls, random drawings are made. After each drawing the ball is returned to the urn, so that each drawing is independent of every other. If the drawings are repeated indefinitely, what is the probability that after some drawing in the sequence an equal number of each of the l kinds of balls will have been drawn?

When an equal number of each of the l kinds of balls have been drawn, we shall say that an equalization occurs. In this paper we shall show that for $l = 2$ or $l = 3$ the probability that an equalization will occur is 1, while for $l \geqq 4$ the probability is less than 1.[1]

2. This problem can be interpreted as a random walk in a network of "streets" in $(l - 1)$-dimensional space. The drawing of a ball of a given kind is represented by the walker's moving a fixed distance in a given direction. Equalization is represented by a return to the origin.

A similar problem has been investigated by G. Pólya [2]: the random walk in a network of "streets" in d-dimensional space where the "streets" are parallel to the coordinate axes. The probability of passing through a given point at least once tends to 1 for $d = 1$ or $d = 2$ but does not tend to 1 for $d \geqq 3$. Since in the present problem the random walk takes place in $(l - 1)$ dimensions, the results are entirely analogous.

For $l = 2$ the random walk occurs on a straight line. The walker moves one unit to the right when a ball of the first kind is drawn and one unit to the left when a ball of the second kind is drawn. It is clear that equalization is represented by a return to the origin.

For $l = 3$ the random walk takes place in a network of streets in a plane. We shall use the complex plane. Let $\omega = \frac{1}{2}(-1 + \sqrt{3}i)$, a cube root of unity.

Represent the drawing of a ball by the addition of 1, ω, or ω^2 according as a ball of the first, second or third kind is drawn. If r balls of the first kind, s of the second kind and t of the third kind have been drawn, the random walker will be at the point

$$r + s\omega + t\omega^2 .$$

If there is a return to the origin,

$$r + s\omega + t\omega^2 = 0 .$$

We know $1 + \omega + \omega^2 = 0$. Therefore, $(r - t) + (s - t)\omega = 0$. It follows that $r = s = t$. Equalization occurs. On the other hand, if equalization occurs,

$$r = s = t , \qquad r + s\omega + t\omega^2 = r\,(1 + \omega + \omega^2) = 0 .$$

[1] It was pointed out to me that certain parts of my result are given in chapter 12 of Professor W. Feller's recent book [1] and that the whole result can be derived by the method given there.