ON SOME CONNECTIONS BETWEEN PROBABILITY THEORY AND DIFFERENTIAL AND INTEGRAL EQUATIONS

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1. Introduction

The connections between probability theory on the one hand and differential and integral equations on the other, are so numerous and diverse that the task of presenting them in a comprehensive and connected manner appears almost impossible.

The aim of this paper is consequently much more modest. We wish mainly to illustrate, on a variety of examples, the interplay between probability problems and the analytic tools used to approach and solve them.

In the traditional approach a probability problem is often reduced to solving a differential or integral equation, and once this has been accomplished we find ourselves outside the field of probability and in a domain where methods of long standing are immediately available. This approach is far from being exhausted. We illustrate it by deriving the so called 'arc sin law' (sections 2 and 3) and certain limiting distributions arising in the study of deviations between theoretical and empirical distributions (section 6). These illustrations are taken from the domain of attraction of the normal law and we are naturally led to the diffusion theory. An attempt to extend the diffusion theory to the case when the normal law is replaced by more general limit laws is made in sections 7 and 8. We are led here to integro-differential equations which offer formidable analytic difficulties and which we were able to solve only in very few cases.

The remainder of the paper is devoted to the reversal of the traditional approach. This reversal consists in an attempt to utilize the probability theory, both rigorously and heuristically, to arrive at results about differential and integral equations. It is in this part that the interplay mentioned above is brought out most clearly.

As an example of the interplay let us mention the following results which are discussed in detail in section 10.

Let Ω be a bounded three dimensional region and let $y \in \Omega$. Let $T \equiv T_\Omega(y)$ be the total time that a Brownian particle starting from y spends in Ω.

It can then be shown that, as $\beta \to \infty$,

$$(1.1) \qquad Pr\{T > \beta\} \sim C(y)\, e^{-\beta/\lambda_1},$$

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