# WIENER'S RANDOM FUNCTION, AND OTHER LAPLACIAN RANDOM FUNCTIONS

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## 1. An application of a formula of Wiener

1.1. Let $X(t)$ be Wiener's well known random function, defined up to an additive constant by the condition

$$X(t) - X(t_0) = \xi \sqrt{t - t_0}, \qquad t > t_0, \tag{1.1.1}$$

$\xi$ being a real and normalized Laplacian (often called Gaussian) random variable. Suppose $0 \leqq t \leqq 2\pi$, $X(0) = 0$, and put

$$X(t) = \frac{t}{2\pi} X(2\pi) + U(t). \tag{1.1.2}$$

The Laplacian function $U(t)$ is completely characterized by its covariance

$$E\{U(t)\, U(t')\} = \frac{u(2\pi - v)}{2\pi} \tag{1.1.3}$$

[$u = \min(t, t')$; $v = \max(t, t')$; $0 \leqq u \leqq v \leqq 2\pi$]. We may conclude that it may be represented by the almost surely convergent Fourier series

$$U(t) = \sum_1^\infty \frac{1}{n\sqrt{\pi}} [\xi_n (\cos nt - 1) + \xi'_n \sin nt], \tag{1.1.4}$$

and that

$$X(t) = \frac{\xi' t}{\sqrt{2\pi}} + \sum_1^\infty \frac{1}{n\sqrt{\pi}} [\xi_n (\cos nt - 1) + \xi'_n \sin nt], \tag{1.1.5}$$

the Greek letters indicating normalized Laplacian random variables, all independent of each other. To prove this, it is sufficient to verify that the Laplacian function (1.1.4) has the covariance (1.1.3).

Thus, the same random function may be defined by (1.1.1) or by (1.1.5). This theorem was proved by N. Wiener [9] in 1924 and, ten years later, formula (1.1.5) was used as a definition by Paley and Wiener. Starting from one or the other point of view, it is easy to prove that $X(t)$ is almost surely a well defined and continuous function; $\delta X(t)$ is generally $O(\sqrt{dt})$ $(dt > 0)$, and not $O(dt)$. Thus $X(t)$ is not differentiable.

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