CONDITIONAL EXPECTATION AND CONVEX FUNCTIONS

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1. Introduction

The *integral* inequality (6) of [1], which extends Blackwell's theorem 2, [2], has in turn been generalized by Hodges and Lehmann, [3, lemma 3.1], who consider a k-dimensional, vector valued function f and replace the absolute s-th power with a real valued, convex function ψ on $\mathcal{E}^k$ (Euclidean k-space). Now, both Blackwell and the present author showed in [1, theorem, p. 281] and in the paragraph following lemma [2, p. 107] that their integral inequalities are consequences of the fact that the *integrands satisfy the same inequalities almost everywhere.* On the other hand, Hodges and Lehmann prove their integral inequality directly (except when conditional probabilities exist almost everywhere as measures), and so there is raised the question of whether or not that inequality is likewise a consequence of an almost everywhere inequality between the integrands. It is the purpose of this note to prove, in the theorem below, that this is indeed the case. Thus, in particular, it is shown that the assertions in [3, section 3] concerning risk functions follow because the same assertions can be made almost everywhere concerning "convex" loss functions.

2. Preliminaries

Let Ω be a space of points x; $\mathcal{Q}$ a σ-field of subsets of Ω; and μ, a probability measure on $\mathcal{Q}$. Let t be a function on Ω to a space Γ of points τ; $\mathcal{T}$, a σ-field of subsets of Γ; and $\mathcal{T}^{-1}$—a sub-σ-field of $\mathcal{Q}$—the inverse of $\mathcal{T}$ under t. Let ν denote the measure on $\mathcal{T}$ defined by $\nu(A) = \mu[t^{-1}(A)]$.

Let f be an $\mathcal{Q}$-measurable, μ-integrable function on Ω to Euclidean k-space, $\mathcal{E}^k$. Finally, let ψ be a (finite) real valued, convex function on $\mathcal{E}^k$. It follows that ψ is continuous [4, p. 19]; we assume that $E[\psi(f)]$ exists.

Conditional expectation with respect to the function t will be indicated in the usual way; for example, $E(f|\cdot)$. Also, let us agree to denote by $[h]_A$ the range of a function h over a subset A of its domain.

The heart of the theorem below is the following

LEMMA. *Let ψ be as above, and let h be a $\mathcal{T}$-measurable function on Γ to $\mathcal{E}^k$. Let $A \in \mathcal{T}$ be a set of positive ν-measure such that for $\tau \in A$ we have $\psi[h(\tau)] > a$.*

Then there exists a $\mathcal{T}$-subset $B \subsetneq A$, with $\nu(B) > 0$, such that for all y in the convex hull of $[h]_B$ we have $\psi(y) > a$.

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