THE ASYMPTOTIC DISTRIBUTION OF CERTAIN CHARACTERISTIC ROOTS AND VECTORS

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1. Introduction

In a number of problems in multivariate statistical analysis use is made of characteristic roots and vectors of one sample covariance matrix in the metric of another. If A^* and D^* are the sample matrices, we are interested in the roots ϕ^* of $|D^* - \phi^* A^*| = 0$ and the associated vectors satisfying $D^* c^* = \phi^* A^* c^*$. In the cases we consider A^* and D^* have independent distributions. Each is distributed like a sum $\sum_{\alpha=1}^{q} y_\alpha y_\alpha'$ where $y_1, \ldots, y_q$ are independently normally distributed with common covariance matrix. In the case of A^* the means of the vectors are zero; in the case of D^* the means may not be zero. We are interested in the asymptotic distribution of the characteristic roots and vectors when the number of vectors defining A^* increases indefinitely and when the means of the vectors defining D^* change in a certain way. The form of the limiting distribution depends on the multiplicity of the roots of a certain determinantal equation involving the parameters. If these roots are simple and different from zero, the asymptotic distribution is joint normal. If the roots are not simple, the asymptotic distribution is expressed in terms of "uniform distributions" on orthogonal matrices and a normal distribution.

We shall first state our problem in a general form and show in what kinds of statistical problems there is interest in these characteristic roots and vectors. Suppose[1] $x_\alpha (\alpha = 1, \ldots, N)$ of p components is normally distributed independently of $x_\beta (\alpha \neq \beta)$ with mean

$$\mathcal{E} x_\alpha = \mathbf{B}_1 z_{1\alpha} + \mathbf{B}_2 z_{2\alpha} \tag{1.1}$$

and covariance

$$\mathcal{E} (x_\alpha - \mathcal{E} x_\alpha)(x_\alpha - \mathcal{E} x_\alpha)' = \mathbf{\Sigma}, \tag{1.2}$$

where $z_{1\alpha}$ and $z_{2\alpha}$ are vectors of fixed variates of q_1 and q_2 components, respectively, and $\mathbf{B}_1$ and $\mathbf{B}_2$ are $p \times q_1$ and $p \times q_2$ matrices, respectively. We shall use the notation $N(\mathbf{B}_1 z_{1\alpha} + \mathbf{B}_2 z_{2\alpha}, \mathbf{\Sigma})$ for the distribution of x_α.

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[1] Unless specifically indicated otherwise, a vector is a column vector; a prime indicates the transpose of a vector or matrix. Vectors and matrices are indicated by bold face type.