SOME APPLICATIONS OF THE CRAMÉR-RAO INEQUALITY

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1. Summary and introduction

In 1945 and 1946, Cramér [1] and Rao [2] independently investigated the problem of obtaining a simple lower bound to the variance of point estimates. In 1947 Wolfowitz [3] simplified the conditions under which Cramér had obtained this bound and extended the result to sequential estimates. In the present paper, use is made of the Cramér-Rao result, in Wolfowitz's form, to investigate some problems of the minimax theory of estimation.

The Bayes method for obtaining minimax estimates developed by Wald since 1939 [4], [5], is completely satisfactory whenever the minimax estimate is the Bayes solution for some *a priori* distribution of the parameter. However, frequently minimax estimates are not Bayes solutions, but only limits of Bayes solutions. When this occurs, the possibility is left open that the minimax estimate is not admissible; that is, that there exists some other minimax estimate whose risk is never greater and is for some parameter value less than that of the given estimate.

In section 2 we consider certain estimation problems in which the loss is proportional to the square of the error of estimate, and use the Cramér-Rao bound to establish directly that certain estimates, which can be shown to be minimax by the Bayes method, are in addition admissible. In section 3 we consider several problems of sequential estimation, for some of which previously no minimax estimates have been known. In all of these cases it turns out that there are minimax estimates based on samples of fixed size.

Problems similar to those treated in the present paper were considered simultaneously by Girshick and Savage [6], the scope of whose work is much larger than ours. Portions of both papers were presented at the joint colloquium of the Stanford and California statistical groups, resulting in a fruitful exchange of ideas. The method introduced here has been employed in [6] to obtain extensions of some of our results.

2. Estimates based on samples of fixed size

Let X be a random variable with distribution P_θ, $\theta \in D$, so that the probability that X falls in a set A is given by

$$P_\theta(A) = \int_A p_\theta(x)\, d\mu(x). \tag{2.1}$$