SOME COMMENTS ON LARGE SAMPLE TESTS

E. L. LEHMANN

UNIVERSITY OF CALIFORNIA, BERKELEY

LET $p_{X|\Theta}$ denote the elementary probability law of a random variable X depending on a parameter Θ, and let $X_1, \ldots, X_n$ be a sample of X. By a test of the hypothesis $H: \Theta = \Theta_0$, we mean a region of rejection W_n in the n-dimensional space of $X_1, \ldots, X_n$. We denote by $P(W_n | \Theta)$ the power of the test region W_n, that is,

$$P(W_n \mid \Theta) = \Pr\left\{\begin{array}{l}\text{rejecting } H \text{ when } \Theta \text{ is the} \\ \text{true value of the parameter.}\end{array}\right\}$$

It is then desired that

$$P(W_n \mid \Theta_0) = \alpha,$$

where α $(0 < \alpha < 1)$ is the preassigned level of significance, and that $P(W_n | \Theta)$ be as close to one as possible for all $\Theta \neq \Theta_0$. If for every other test Z_n satisfying $P(Z_n | \Theta_0) = \alpha$ we have $P(W_n | \Theta) \geqq P(Z_n | \Theta)$ for all Θ, W_n is said to be uniformly most powerful. In the theory of large-sample tests, sequences of tests $\{W_n\}$, $\{Z_n\}$, $(n = 1, 2, \ldots)$, are compared on the basis of asymptotic properties of the power functions $P(W_n | \Theta)$, $P(Z_n | \Theta)$.

Various classes of asymptotically "good" or "best" tests have been defined. Among them: Tests "unbiased in the limit" **[1]**,[1] "asymptotically most powerful tests" **[2]**, and "asymptotically most stringent tests **[3]**. As was pointed out by Wald (**[4]**, p. 32) concerning his asymptotically most powerful (AMP) tests, these definitions specify that the tests are powerful for sufficiently large n, but say little or nothing about the speed with which the limiting power is approached.

The purpose of the present note is to illustrate this remark by two examples. In example 1 an AMP test sequence W_n is constructed and a sequence of tests Z_n, corresponding to the same level of significance, such that for all Θ

$$\frac{1 - P(Z_n \mid \Theta)}{1 - P(W_n \mid \Theta)} \to 0 \text{ as } n \to \infty.$$

Example 2 shows that, given any sequence of numbers $r_n \to 1$, there exists an AMP test W_n such that $P(W_n \mid \Theta) \leqq r_n$ for all Θ.

The tests W_n of examples 1 and 2 are also asymptotically most stringent, and similar tests exist which are unbiased in the limit. Actually it seems doubtful that any definition of optimum tests, based only on asymptotic

[1] Boldface numbers in brackets refer to references at the end of the paper (p. 457).