TIME SERIES AND HARMONIC ANALYSIS*

J. L. DOOB

UNIVERSITY OF ILLINOIS

Introduction

Although many articles on the present subject have appeared in the mathematical, statistical, and physical literature, there still seems to be some justification for one more. The statisticians have applied only small parts of the theory; the physicists have gone deeper, but write like physicists; the mathematicians have gone furthest, but write like mathematicians, only for posterity. Their work is frequently not understood, and is in general either ignored or applied in simplified forms which often are formally more formidable than the original rigorous one. The present paper attempts to give a compact outline of the harmonic analysis of stochastic processes, with applications to physical problems.*

Time series can be analyzed from two points of view.

a) A time series is a sequence of numbers, to be analyzed for trend, periodicity, prediction possibilities, and the like. The source of the series, that is, the mathematics of the background of the numbers, is ignored.

b) A time series is a sequence of numbers arising from a certain function $f(t)$ (where t is time) as t takes on a sequence of values. There are ordinarily probability parameters in $f(t)$, so that the function value $f(t_0)$ is not uniquely determined by t_0; the function values obtained determine a sample function of a stochastic process. The general properties of this process are deduced from an analysis of the origin of the series, helped by the actual sample values obtained. Trends, periods, and the like are determined in terms of the properties of the stochastic process.

Although the first point of view is superficial, it is adequate in many applications. One reason for this adequacy is the parallelism between the properties of a stochastic process (that is, the average properties of its sample functions) and the properties of almost all the individual sample functions. The harmonic analysis of an individual sample function is formally almost identical with that of the process. In other terms, the formal analysis applied to specific data is largely independent of the background, and is essentially equivalent to that applied to stochastic processes. Although the main part of this paper

* *Note added in proofs, August 10, 1948:* Since the present paper was written, almost three years ago, a number of contributions have appeared in print, particularly by French and Scandinavian writers. Because of these publications a complete reorganization of the paper might be in order. However, it was thought best to leave the paper in its original form although it has lost much of its freshness since it was written. This explains the omission of references to the work of Loève, Karhunen, and others.