REMARKS ON CHARACTERISTIC FUNCTIONS

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Introduction

This short paper consists of two parts which have little in common except that in both we discuss characteristic functions of one-dimensional probability distributions. In the first part we consider characteristic functions of a certain special type whose principal merit lies in the fact that it is easily recognizable. In the second part we deal with finite distributions (contained in a certain finite interval) and with finitely different distributions (coinciding outside a certain finite interval).

The notation follows that of Cramér's well-known tract.[1] A distribution function is denoted by a capital letter, as $F(x)$, and the corresponding characteristic function by the corresponding small letter, as $f(t)$. Thus

$$f(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x)\,. \tag{1}$$

$F(x)$ is real-valued, never decreasing; $F(-\infty) = 0$; $F(\infty) = 1$. Therefore

$$f(0) = 1\,. \tag{2}$$

Moreover $f(t)$ is continuous for all real values of t and has the properties

$$|f(t)| \leqq 1\,, \tag{3}$$

$$f(-t) = \overline{f(t)}\,, \tag{4}$$

that is, $f(-t)$ and $f(t)$ are conjugate complex.

I. A Simple Type of Characteristic Functions

1. A Sufficient Condition for Characteristic Functions

We are given a function, defined for all real values of t; is it the characteristic function of some probability distribution? This question is often important but not often easy to answer. The properties mentioned in the introduction [relations (2), (3), (4), and continuity] constitute simple *necessary* conditions that a given function $f(t)$ should be characteristic. Yet these conditions, taken together, are far from being sufficient. *Necessary and sufficient*

[1] Cramér [**3**]. Boldface numbers in brackets refer to references at the end of the paper (see p. 123).